NATIONAL TECHNICAL UNIVERSITY OF ATHENS (NTUA)
SCHOOL OF MECHANICAL ENGINEERING
PARALLEL CFD & OPTIMIZATION UNIT (PCOpt/NTUA)

Constrained Optimisation. The KKT Conditions

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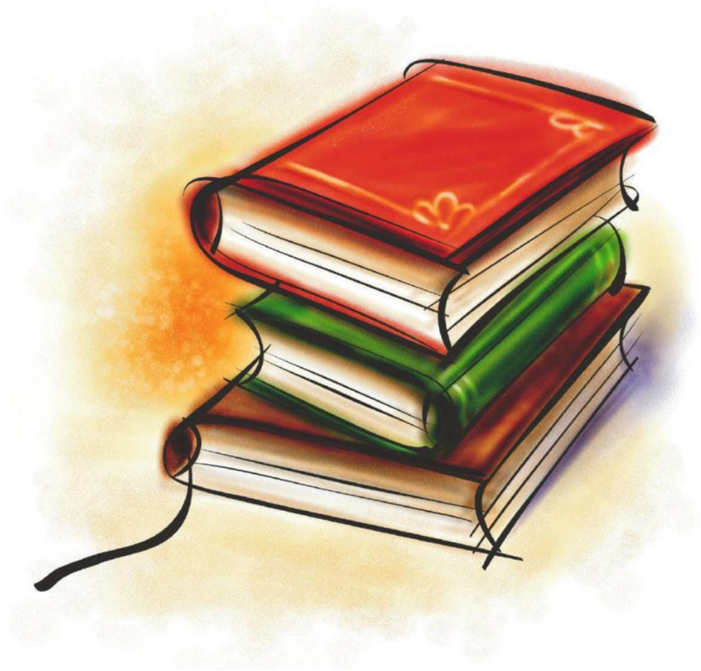


Constrained Optimisation - Background

Type of Constraints in an Optimisation Problem

Design or optimisation variables' bounds:

$$x_i^L \leq x_i \leq x_i^U$$

Inequality Constraints:

$$c_i(\vec{x}) \leq 0 \qquad i = 1, \ldots, K_2 \qquad i \in I$$

Equality Constraints:

$$c_i(\vec{x}) = 0 \qquad i = K_2 + 1, \ldots, K_1 + K_2 \qquad i \in E$$

Feasible Space:

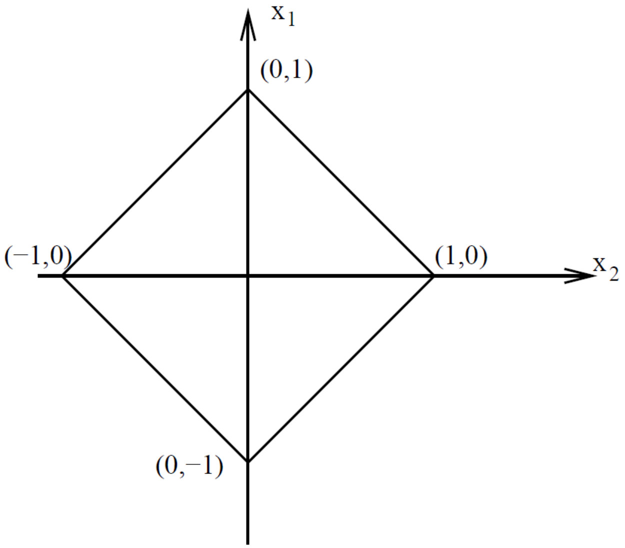
$$\Omega = \{\vec{x} \mid c_i(\vec{x}) \leq 0, \quad i \in I; c_i(\vec{x}) = 0, \quad i \in E\}$$

$$\boxed{\min_{\vec{x} \in \Omega} \quad F(\vec{x})}$$

Constrained Optimisation – Interesting Examples

Example 1:

$$|x_1| + |x_2| \le 1 \quad \Rightarrow \quad \left\{ \begin{array}{rcl} x_1 + x_2 & \le & 1 \\ x_1 - x_2 & \le & 1 \\ -x_1 + x_2 & \le & 1 \\ -x_1 - x_2 & \le & 1 \end{array} \right.$$



Example 2:

$$\min_{\vec{x} \in \Re} F(\vec{x}) = \min_{\vec{x} \in \Re} max(x^2, x) \quad \Rightarrow \quad \left\{ \begin{array}{c} \min\limits_{t \in \Re, x \in \Re,} \quad t \\ \\ x \le t \quad , \quad x^2 \le t \end{array} \right.$$

J.Nocedal, S.Wright, Numerical Optimization, Springer,1999.

Minimisation using a Single Equality Constraint

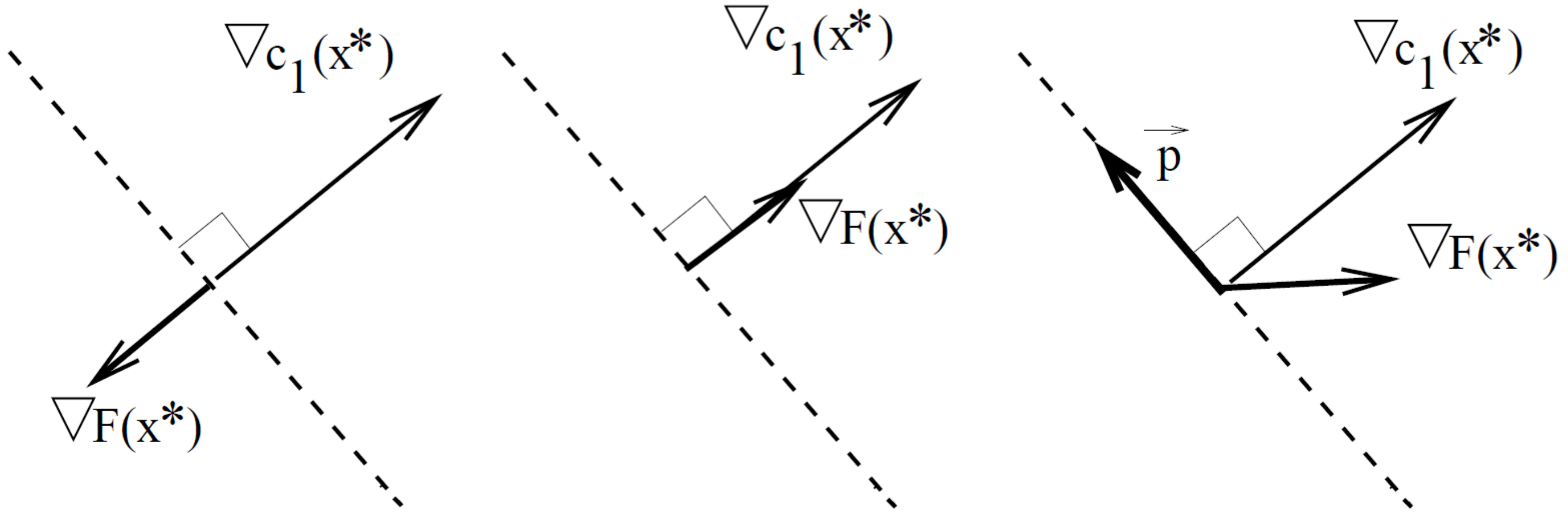
$$\min_{\vec{x}\in\Re^N} F(\vec{x})\,, \qquad \models \qquad c_1(\vec{x}) = 0$$

The necessary condition for $\vec{x}^*$ to be the optimal solution is that there exists no direction $\vec{p}$ satisfying both equations:

$$\left.\begin{array}{l} \vec{p}^{\,T}\,\nabla c_1(\vec{x}^*) = 0 \\ \vec{p}^{\,T}\,\nabla F(\vec{x}^*) < 0 \end{array}\right\}$$

Minimisation using a Single Equality Constraint

$$\min_{\vec{x} \in \Re^N} F(\vec{x}) \,, \qquad \models \qquad c_1(\vec{x}) \;=\; 0$$



$$\left.\begin{aligned} \vec{p}^{\,T}\, \nabla c_1(\vec{x}^{\,*}) &= 0 \\ \vec{p}^{\,T}\, \nabla F(\vec{x}^{\,*}) &< 0 \end{aligned}\right\}$$

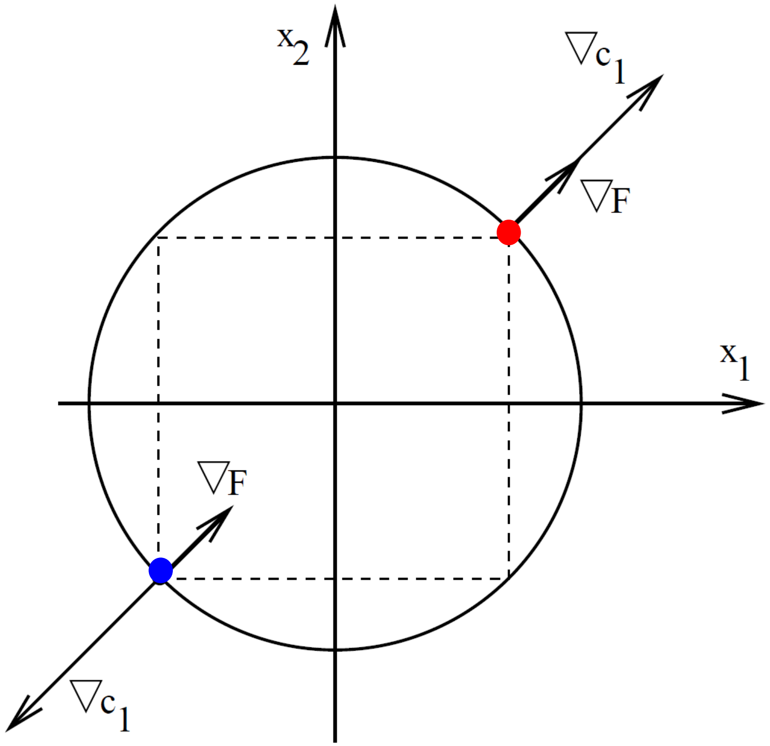
A direction satisfying both equations is:

$$\vec{p} \;=\; -\left(I - \frac{\nabla c_1(\vec{x}^{\,*})^T \nabla c_1(\vec{x}^{\,*})}{||\nabla c_1(\vec{x}^{\,*})||^2}\right) \nabla F(\vec{x}^{\,*})$$

Minimisation using a Single Equality Constraint – Example:

$$\min_{\vec{x}\in\Re^2} F(\vec{x})\;, \qquad F(\vec{x}) = x_1 + x_2$$

$$\models \qquad c_1(\vec{x}) \;=\; x_1^2 + x_2^2 - 2 = 0$$



J.Nocedal, S.Wright, Numerical Optimization, Springer,1999.

Minimisation using a Single Equality Constraint – Lagrangian Function

Consider the same problem using a Lagrangian function L:

$$\min_{\vec{x}\in\Re^N} F(\vec{x}), \qquad \models \qquad c_1(\vec{x}) = 0$$

$$L(\vec{x},\lambda_1) = F(\vec{x}) - \lambda_1 c_1(\vec{x})$$

$$\nabla L(\vec{x},\lambda_1) = \nabla F(\vec{x}) - \lambda_1 \nabla c_1(\vec{x}) = 0$$

At the optimal solution $\vec{x}^*$, there should be a scalar (Lagrange multiplier) λ_1^* zeroing $\nabla F(\vec{x}^*)$ or:

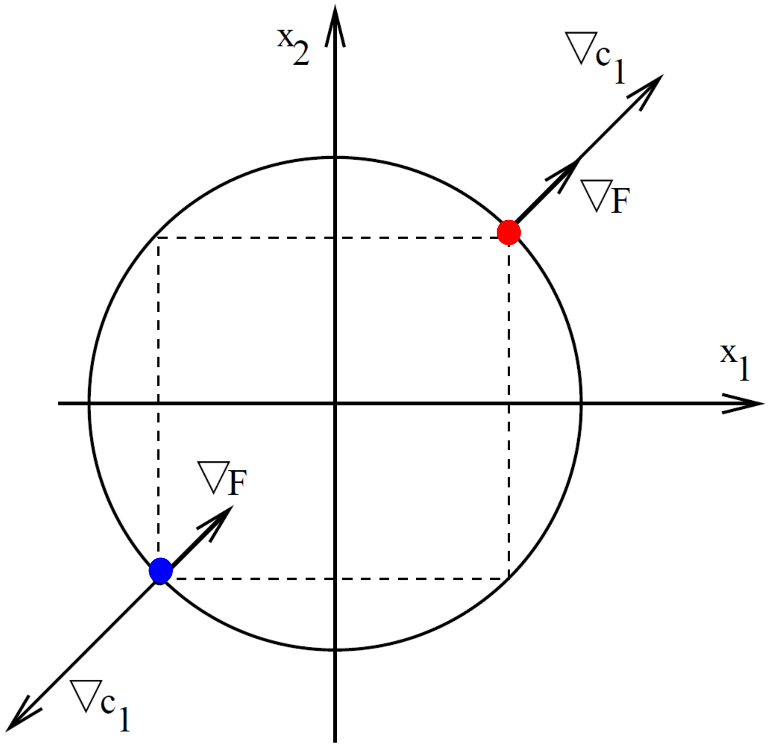
$$\nabla F(\vec{x}^*) = \lambda_1^* \nabla c_1(\vec{x}^*)$$

Feasible Space

Revisited:

$$\min_{\vec{x} \in \Re^2} F(\vec{x}) \,, \qquad F(\vec{x}) = x_1 + x_2$$

$$\models \qquad c_1(\vec{x}) \;=\; x_1^2 + x_2^2 - 2 = 0$$



Necessary Condition:

$$\boxed{\begin{array}{c} \nabla F(\vec{x}^*) \;=\; \lambda_1^* \, \nabla c_1(\vec{x}^*) \\ \lambda_1^* \text{ unrestricted in sign!} \end{array}}$$

J.Nocedal, S.Wright, Numerical Optimization, Springer,1999.

Minimisation using a Single Inequality Constraint

$$\min_{\vec{x} \in \Re^N} F(\vec{x}) \,, \qquad \models \qquad c_1(\vec{x}) \leq 0$$

The necessary condition for $\vec{x}^*$ to be the optimal solution is:

$$\nabla F(\vec{x}^*) = \lambda_1^* \nabla c_1(\vec{x}^*) \,, \qquad \lambda_1^* \leq 0$$

and:

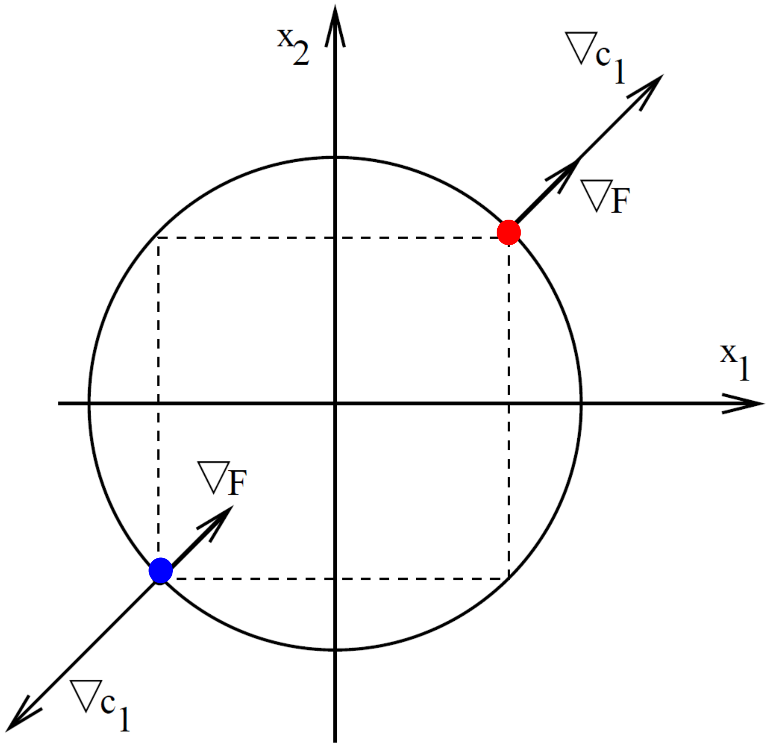
$$\lambda_1^* c_1(\vec{x}^*) = 0$$

Minimisation using a Single Inequality Constraint – Example:

$$\min_{\vec{x}\in\Re^2} F(\vec{x})\,, \qquad F(\vec{x}) = x_1 + x_2$$

$$\models \qquad c_1(\vec{x}) \;=\; x_1^2 + x_2^2 - 2 \;\leq\; 0$$

$$\boxed{\begin{gathered}\nabla F(\vec{x}^*) \;=\; \lambda_1^* \,\nabla c_1(\vec{x}^*)\,, \qquad \lambda_1^* \leq 0 \\ \lambda_1^* c_1(\vec{x}^*) = 0\end{gathered}}$$



J.Nocedal, S.Wright, Numerical Optimization, Springer,1999.

First-Order (Necessary) Optimality Conditions

Problem:

$$F(\vec{x}),\ F : \Re^N \rightarrow \Re$$
$$c_i(\vec{x}) \leq 0 \qquad i \in I$$
$$c_i(\vec{x}) = 0 \qquad i \in E$$

Lagrangian Function::

$$L(\vec{x}, \vec{\lambda}) = F(\vec{x}) - \sum_{i \in E \cup I} \lambda_i c_i(\vec{x})$$

Necessary Optimality Conditions:

$$\begin{aligned} \nabla L(\vec{x}^*, \vec{\lambda}^*) &= 0 \\ c_i(\vec{x}^*) &= 0, \quad \forall\, i \in E \\ c_i(\vec{x}^*) &\leq 0, \quad \forall\, i \in I \\ \lambda_i^* &\leq 0, \quad \forall\, i \in I \\ \lambda_i\, c_i(\vec{x}^*) &= 0, \quad \forall\, i \in E \cup I \end{aligned}$$

or

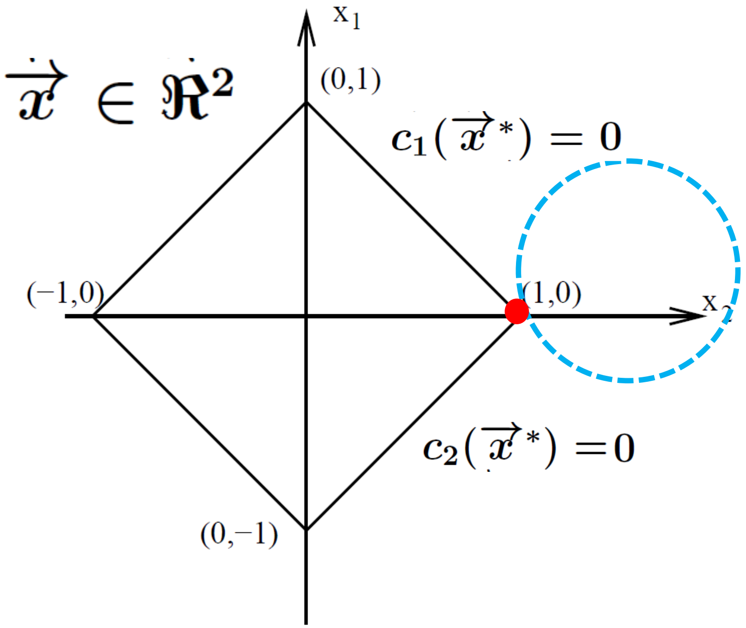
$$\nabla F(\vec{x}^*) - \sum_{i \in E \cup I} \lambda_i \nabla c_i(\vec{x}) = 0$$

Karush-Kuhn-Tucker (KKT) Conditions

First-Order (Necessary) Optimality Conditions - Example

$$\begin{aligned} F(x_1,x_2) &= (x_1-\frac{3}{2})^2+(x_2-\frac{1}{8})^2 \\ \models c_1(x_1,x_2) &= x_1+x_2-1\le 0 \\ c_2(x_1,x_2) &= x_1-x_2-1\le 0 \\ c_3(x_1,x_2) &= -x_1+x_2-1\le 0 \\ c_4(x_1,x_2) &= -x_1-x_2-1\le 0 \end{aligned}$$

$\vec{x}\in\Re^2$



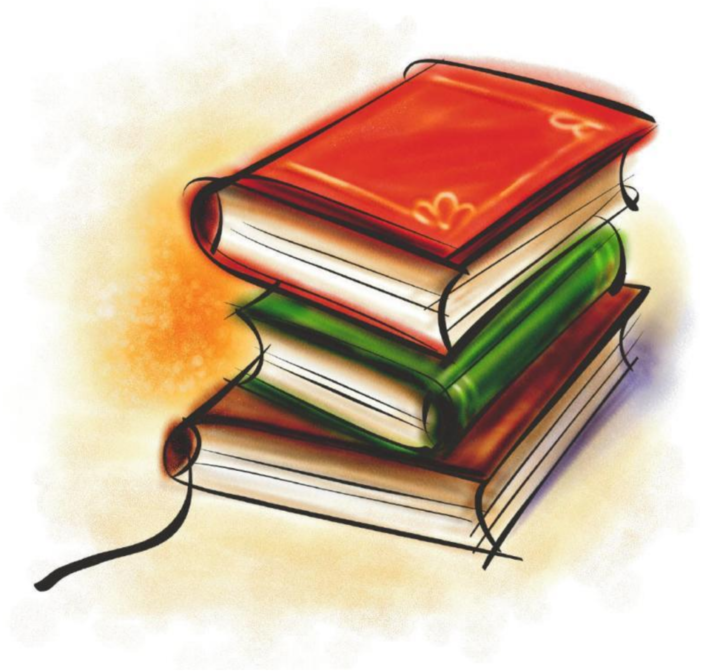
Optimal Solution: $\vec{x}^*=(1,0)$

$$\begin{aligned} \nabla F(\vec{x}^*) &= (-1\,,\,-\frac{1}{2})^T \\ \nabla c_1(\vec{x}^*) &= (1\,,\,1)^T \\ \nabla c_2(\vec{x}^*) &= (1\,,\,-1)^T \end{aligned}$$

$$\boxed{(\lambda_1^*,\lambda_2^*,\lambda_3^*,\lambda_4^*) = (-\frac{3}{4},-\frac{1}{4},0,0)}$$

$c_3(\vec{x}^*)=-2<0 \qquad c_4(\vec{x}^*)=-2<0$

J.Nocedal, S.Wright, Numerical Optimization, Springer,1999.



Possible Ways to Solve Constrained Optimisation Problems using a GBM

Sequential Unconstrained Minimisation Techniques (SUMT)

$$min \quad F \quad (\vec{x})$$
$$\models \quad c_i(\vec{x}) \leq 0\,, \quad i \in I$$
$$c_i(\vec{x}) = 0\,, \quad i \in E$$

Pseudo-Objective Function:

$$\Phi(\vec{x}, w_p) \; = \; F(\vec{x}) \; + \; w_p P(\vec{x})$$

$P(\vec{x}) \; = \; 0$ **at all feasible solutions.**

Different ways to define the Penalty Function P( ) ….

G. Vanderplaats, McGraw-Hill Book Company,1976.

SUMT: The Exterior Penalty Method (1/3)

$$P(\vec{x}) = \sum_{i\in I}[max(0, c_i(\vec{x}))]^2 + \sum_{i\in E}[c_i(\vec{x})]^2$$

w_p
- :small value → Φ is easily minimised but constraint violation might occur.
- :large value → easier to satisfy the constraints; convergence issues.

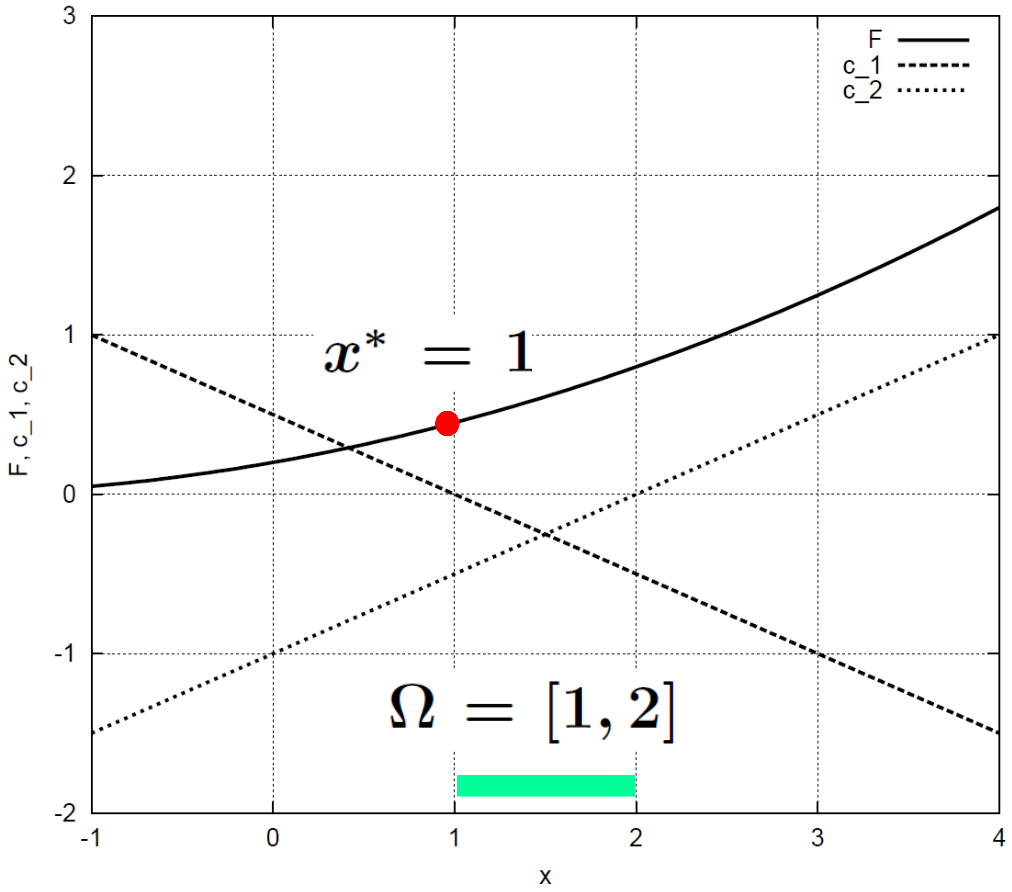
G. Vanderplaats, McGraw-Hill Book Company,1976.

SUMT: The Exterior Penalty Method – Example (2/3)

$$min \quad F \quad (x) = \frac{(x+2)^2}{20}$$

$$\models \quad c_1(x) = \frac{1-x}{2} \leq 0$$

$$c_2(x) = \frac{x-2}{2} \leq 0$$



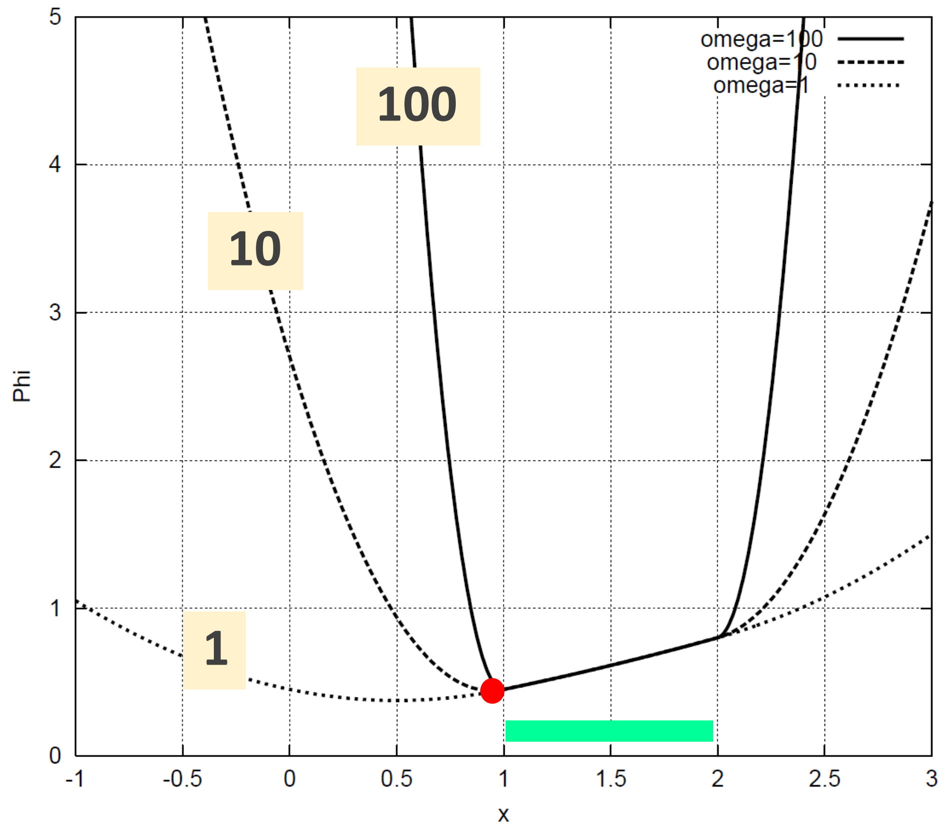
G. Vanderplaats, McGraw-Hill Book Company,1976.

SUMT: The Exterior Penalty Method – Example (3/3)

$$\Phi(\vec{x}, w_p) = F(\vec{x}) + w_p P(\vec{x})$$

Programming tips: start with a small w_p value and increase it (e.g. *2 or *3) in each optimisation cycle.

In any premature termination, the optimal solution is unfeasible!



$$(w_p = 100, 10, 1)$$

G. Vanderplaats, McGraw-Hill Book Company,1976.

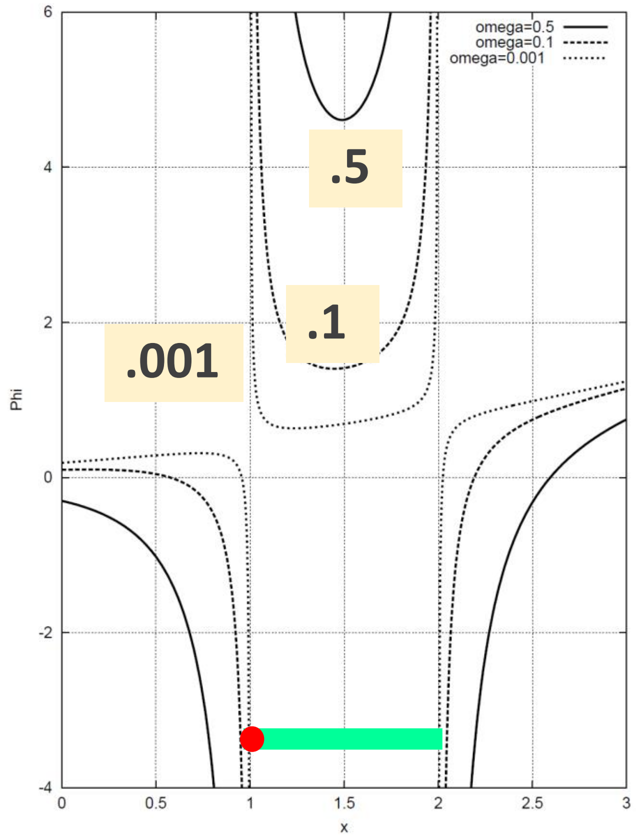
SUMT: The Interior Penalty Method (1/2)

$$P(\vec{x}) = \sum_{i \in I} \frac{-1}{c_i(\vec{x})} \qquad P(\vec{x}) = \sum_{i \in I} [-ln(-c_i(\vec{x}))]$$

$$\Phi(\vec{x}, w_p) = F(\vec{x}) + w'_p P(\vec{x}) + w_p \sum_{i \in E} [c_i(\vec{x})]^2$$

Programming tips: start with a high w'_p value and decrease it during the optimisation loop.

$$w'_p \rightarrow 0$$
$$w_p \rightarrow \infty$$



$$w'_p = 0.5, 0.1, 0.01$$

G. Vanderplaats, McGraw-Hill Book Company,1976.

SUMT: The (Extended) Interior Penalty Method (2/2)

$$\Phi(\vec{x}, w_p) = F(\vec{x}) + w'_p P(\vec{x}) + w_p \sum_{i \in E} [c_i(\vec{x})]^2$$

$$P(\vec{x}) = \sum_{i \in I} \tilde{c}(\vec{x})$$

$$\tilde{c}(\vec{x}) = -\frac{1}{c_i(\vec{x})} \quad , \quad if \;\; c_i(\vec{x}) \leq \varepsilon$$

$$\tilde{c}(\vec{x}) = -\frac{2\varepsilon - c_i(\vec{x})}{\varepsilon^2} \quad , \quad if \;\; c_i(\vec{x}) > \varepsilon$$

ε is a small negative number.

G. Vanderplaats, McGraw-Hill Book Company,1976.

The Augmented Lagrange Multiplier (ALM) Method (1/2)

$$\min_{\vec{x} \in \Re^N} \; F(\vec{x}) \,, \qquad \models \qquad c_i(\vec{x}) \;=\; 0 \,, \qquad i \in E$$

$$L(\vec{x}, \vec{\lambda}) \;=\; F(\vec{x}) - \sum_{i \in E} \lambda_i c_i(\vec{x})$$

Use the KKT conditions and the exterior penalty method to minimise the pseudo-objective function:

$$\Phi^*(\vec{x}, \vec{\lambda}, w_p) \;=\; F(\vec{x}) + \sum_{i \in E} \left(-\lambda_i c_i(\vec{x}) + w_p [c_i(\vec{x})]^2\right)$$

G. Vanderplaats, McGraw-Hill Book Company,1976.

The Augmented Lagrange Multiplier (ALM) Method (2/2)

$$\Phi^*(\vec{x}, \vec{\lambda}, w_p) = F(\vec{x}) + \sum_{i \in E} \left(-\lambda_i c_i(\vec{x}) + w_p [c_i(\vec{x})]^2\right)$$

Step 1: Initialise x, λ, ω_P, its multiplier γ and the max. value $\omega_{P,max}$.

Step 2: Minimise Φ^* by solving an unconstrained minimisation problem.

Step 3: Modify

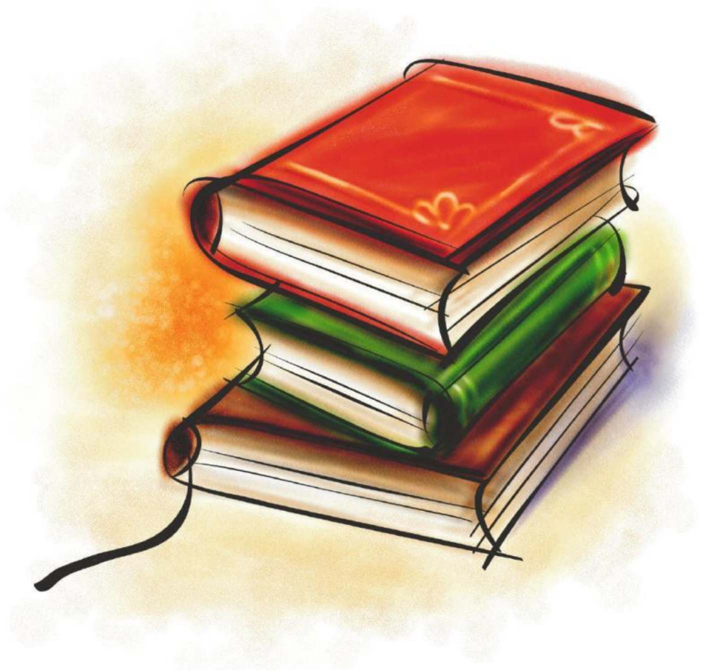
$$\lambda_i \leftarrow \lambda_i - 2 w_p c_i(\vec{x}), \qquad i \in E$$

$$w_p \leftarrow min(\gamma w_p, w_p^{max})$$

Return to Step 1.

This can be modified for problems with inequality constraints…..

G. Vanderplaats, McGraw-Hill Book Company,1976.



Sequential Quadratic Programming (SQP)

Sequential Quadratic Programming (SQP) (1/3)

Consider a minimisation problem with N unknowns and M equality-constraints:

$$\begin{cases} \min f(x) & x \in R^N \\ s.t.\ c(x) = 0 & c \in R^M \end{cases}$$

Lagrangian function:

$$L(x,\lambda) = f(x) - \lambda^{\mathrm{T}} c(x) \qquad \lambda \in R^M$$

KKT conditions, i.e. the necessary conditions for optimality (derivatives w.r.t. x and λ) (N+M eqs.):

$$F(x,\lambda) = \begin{bmatrix} \nabla f(x) - A(x)^{\mathrm{T}}\lambda \\ c(x) \end{bmatrix} = 0$$

where the transposed Jacobian matrix of the constraints is:

$$A(x)^{\mathrm{T}} = [\nabla c_1(x) \quad \nabla c_2(x) \quad \dots \quad \nabla c_m(x)]$$

Sequential Quadratic Programming (SQP) (2/3)

Solution of the Newton's method (k=iteration index):

$$\begin{bmatrix} x_{k+1} \\ \lambda_{k+1} \end{bmatrix} = \begin{bmatrix} x_k \\ \lambda_k \end{bmatrix} + \begin{bmatrix} p \\ q \end{bmatrix}$$

$p \in R^N$

$q \in R^M$

where p,q are the Newton's steps:

$$\underbrace{\begin{bmatrix} W_k & -A_k^T \\ A_k & 0 \end{bmatrix}}_{\text{KKT Matrix}} \begin{bmatrix} p \\ q \end{bmatrix} = -\begin{bmatrix} \nabla f_k - A_k^T \lambda_k \\ c_k \end{bmatrix}$$

Reminder:

$$L(x,\lambda) = f(x) - \lambda^{\mathrm{T}} c(x)$$

$$F(x,\lambda) = \begin{bmatrix} \nabla f(x) - A(x)^{\mathrm{T}} \lambda \\ c(x) \end{bmatrix} = 0$$

and $W(x,\lambda) = \nabla_{xx}^2 L(x,\lambda)$.

The efficient application of the Newton's method requires the efficient solution of the linear system, by exploiting its block 2x2 structure, the possible use of Quasi-Newton methods (approximate rather than solving the exact Hessian matrix) and usually needs a good initialiσation of λ.

Sequential Quadratic Programming (SQP) (3/3)

Instead, one may define the quadratic problem:

$$\min \left[\frac{1}{2}p^T W_k p - \nabla f_k^T\, p\right] \qquad s.t. \qquad A_k p + c_k = 0 \qquad \begin{array}{l} p \in R^N \\ q \in R^M \end{array}$$

where p, q are constant vectors; the objective function is quadratic in p, whereas the constraints are linear in p. Its Lagrangian is expressed as:

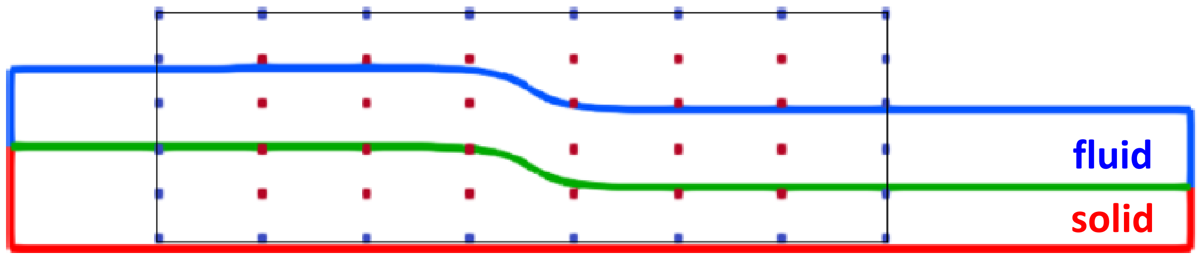
$$\mathrm{L} = \frac{1}{2}p^T W_k p - \nabla f_k^T\, p - \mu^{\mathrm{T}}(A_k p + c_k)$$

We can solve this problem by calculating the setting the gradient of the new Lagrangian L to zero! It can easily be seen that we get the exact same system so solve as in the previously presented Newton's method at step k. This means that, in Newton's method, at each step k, we exactly solve a minimisation problem with a quadratic objective function and linear constraints! This is usually refereed to as a quadratic program! This is the essence of the SQP method.

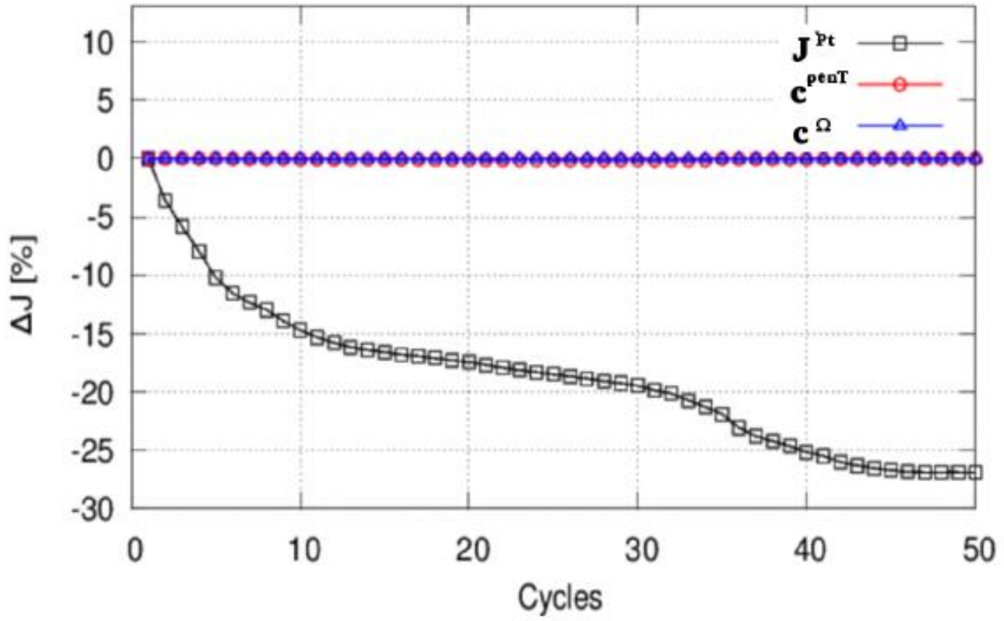
J.Nocedal, S.Wright, Numerical Optimization, Springer,1999.

Constrained ShpO of a Cooling Channel using SQP (1/2)

CHT problem: T^F =291.2 K, T^S =500 K.
Objective: min. total pressure losses (J^{pt})
st. a) same solid volume
b) max. T^S does not increase

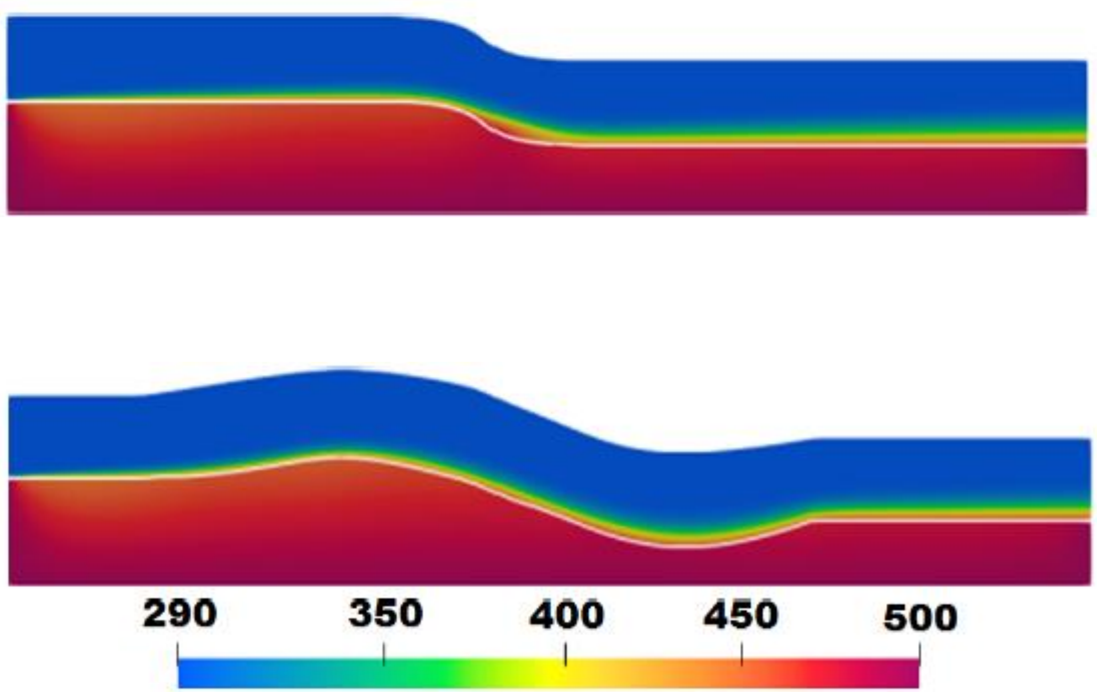


ΔJ^{pt} =-26.9%
Constraints satisfied

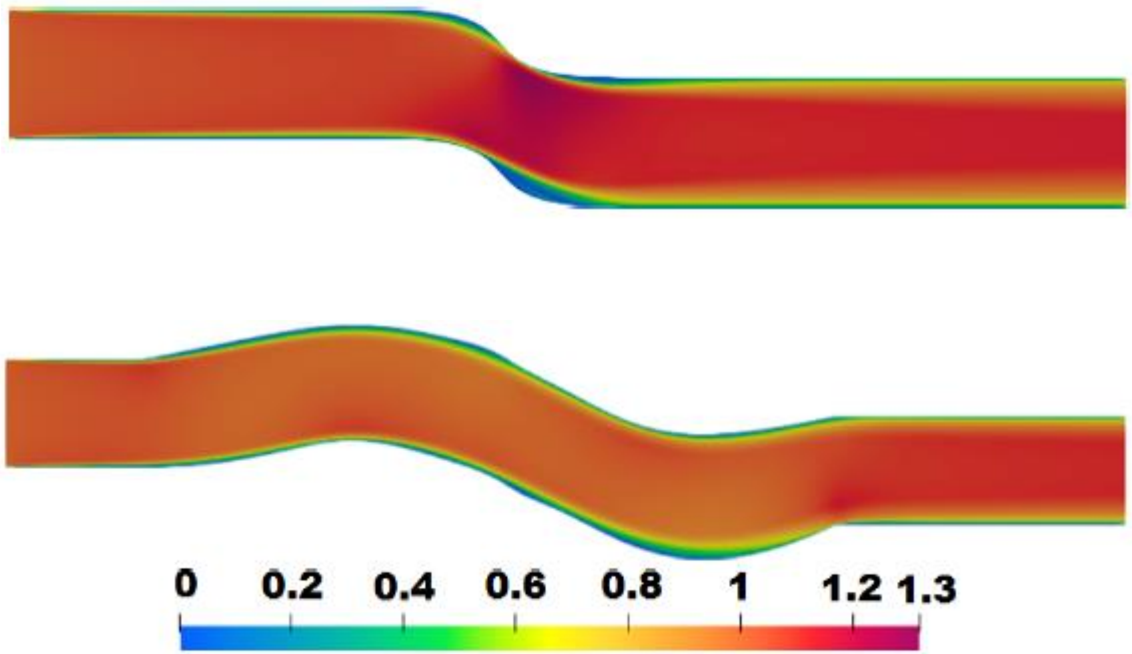


Constrained ShpO of a Cooling Channel using SQP (2/2)

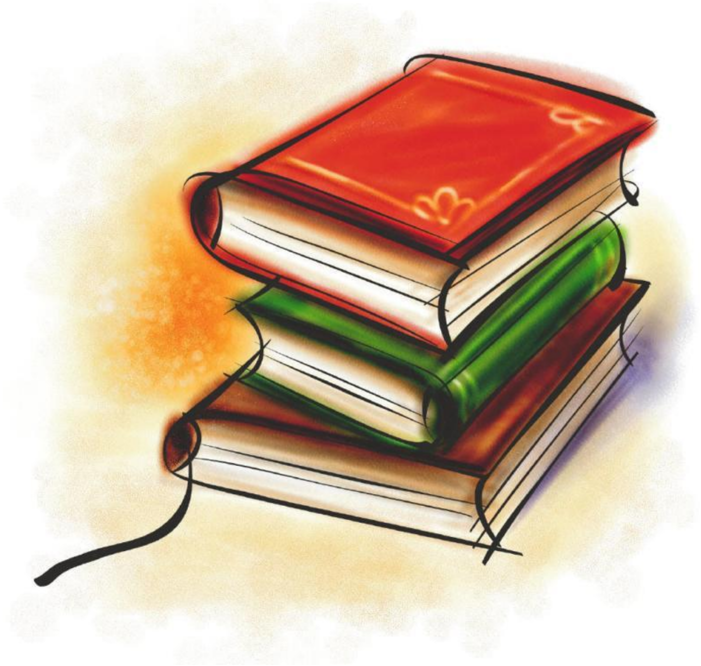
Temperature field



Velocity field



****x,y axes not in scale**



The Method of Moving Asymptotes (MMA)

Constrained SOO Problem Definition

Minimise: $f_0(\vec{b})$

subject to:

$$\begin{cases} f_i(\vec{b}) \leq 0, & i = 1, ..., M_c \\ b_j^{min} \leq b_j \leq b_j^{max}, & j = 1, ..., N \end{cases}$$

Minimisation using a special type of convex optimisation; initially proposed only for structural sizing problems.

In each step of the optimisation algorithm, a convex approximating sub-problem is formulated and solved.

Questions:

1. **How these approximating functions are defined?**
2. **How each sub-problem can be solved?**

Local Problem Definition (at cycle (k))

At the (k)-th cycle, define:

$$f_i^{(k)}(\vec{b}) = r_i^{(k)} + \sum_{j=1}^{N}\left(\frac{p_{ij}^{(k)}}{U_j^{(k)} - b_j} + \frac{q_{ij}^{(k)}}{b_j - L_j^{(k)}}\right), \quad i = 0, 1, \dots, M_c$$

where:

$$r_i^{(k)} = f_i(\vec{b}^{(k)}) - \sum_{j=1}^{N}\left(\frac{p_{ij}^{(k)}}{U_j^{(k)} - b_j^{(k)}} + \frac{q_{ij}^{(k)}}{b_j^{(k)} - L_j^{(k)}}\right), \quad i = 0, 1, \dots, M_c$$

and:

$$p_{ij}^{(k)} = (U_j^{(k)} - b_j^{(k)})^2 \frac{\partial f_i}{\partial b_j}, \quad if \quad \frac{\partial f_i}{\partial b_j} > 0$$

$$p_{ij}^{(k)} = 0, \quad if \quad \frac{\partial f_i}{\partial b_j} \leq 0$$

$$q_{ij}^{(k)} = 0, \quad if \quad \frac{\partial f_i}{\partial b_j} \geq 0$$

$$q_{ij}^{(k)} = -(b_j^{(k)} - L_j^{(k)})^2 \frac{\partial f_i}{\partial b_j}, \quad if \quad \frac{\partial f_i}{\partial b_j} < 0$$

Local Problem Definition (at cycle (k))

Properties:

$$f_i(\vec{b}^{(k)}) = f_i^{(k)}(\vec{b}^{(k)}), \quad i = 0, ..., M_c$$

$$\frac{\partial f_i(\vec{b}^{(k)})}{\partial b_j} = \frac{\partial f_i^{(k)}(\vec{b}^{(k)})}{\partial b_j}, \quad i = 0, ..., M_c, \quad j = 1, ..., N$$

Why is this a Convex Problem?

$$\frac{\partial^2 f_i}{\partial b_j^2} = \frac{2p_{ij}^{(k)}}{(U_j^{(k)} - b_j^{(k)})^3} + \frac{2q_{ij}^{(k)}}{(b_j^{(k)} - L_j^{(k)})^3}$$

$$\frac{\partial^2 f_i}{\partial b_j \partial b_m} = 0, \quad j \neq m$$

where: $p_{ij}^{(k)} \geq 0, \quad q_{ij}^{(k)} \geq 0$

Selection of the Moving Asymptotes

$k=0, k=1:$

$$L_j^{(k)} = b_j^{(k)} - \Delta b_j, \quad U_j^{(k)} = b_j^{(k)} + \Delta b_j$$

$$\Delta b_j = b_j^{max} - b_j^{min}, \quad j=1,...,N$$

$k \geq 2:$

- $if \quad (b_j^{(k)} - b_j^{(k-1)})(b_j^{(k-1)} - b_j^{(k-2)}) < 0:$

$$\begin{aligned} L_j^{(k)} &= b_j^{(k)} - s(b_j^{(k-1)} - L_j^{(k-1)}) \\ U_j^{(k)} &= b_j^{(k)} + s(L_j^{(k-1)} - b_j^{(k-1)}) \end{aligned}$$

- $if \quad (b_j^{(k)} - b_j^{(k-1)})(b_j^{(k-1)} - b_j^{(k-2)}) > 0:$

$$\begin{aligned} L_j^{(k)} &= b_j^{(k)} - (b_j^{(k-1)} - L_j^{(k-1)})/s \\ U_j^{(k)} &= b_j^{(k)} + (L_j^{(k-1)} - b_j^{(k-1)})/s \end{aligned}$$

where:
$s = 0.7$
or so……

Redefine the Problem:

Minimise:

$$f_0^{(k)}(\vec{b}) = r_0^{(k)} + \sum_{j=1}^{N} \left(\frac{p_{0j}^{(k)}}{U_j^{(k)} - b_j} + \frac{q_{0j}^{(k)}}{b_j - L_j^{(k)}} \right)$$

subject to:

$$\begin{cases} f_i^{(k)}(\vec{b}) = r_i^{(k)} + \sum_{j=1}^{N} \left(\frac{p_{ij}^{(k)}}{U_j^{(k)} - b_j} + \frac{q_{ij}^{(k)}}{b_j - L_j^{(k)}} \right) \le 0, \quad i = 1, \ldots, M_c \\ max(b_j^{min}, \alpha_j^{(k)}) \le b_j \le min(b_j^{max}, \beta_j^{(k)}), \quad j = 1, \ldots, N \end{cases}$$

where, for instance:

$$\begin{aligned} \alpha_j^{(k)} &= 0.90 L_j^{(k)} + 0.10 b_j^{(k)} \\ \beta_j^{(k)} &= 0.90 U_j^{(k)} + 0.10 b_j^{(k)} \end{aligned}$$

Solve it using a Lagrangian Function

$$\mathcal{L}(\vec{b},\vec{\Psi}) = f_0^{(k)}(\vec{b}) + \sum_{i=1}^{M_c} \Psi_i f_i^{(k)}(\vec{b}) = f_0^{(k)}(\vec{b}) + \vec{\Psi}^T \vec{f}_{con}^{(k)}(\vec{b})$$

$$\mathcal{L}(\vec{b},\vec{\Psi}) = r_0^{(k)} + \vec{\Psi}^T \vec{r}_{con}^{(k)} + \sum_{j=1}^{N} \underbrace{\left(\frac{p_{0j}^{(k)} + \vec{\Psi}^T \vec{p}_j^{(k)}}{U_j^{(k)} - b_j} + \frac{q_{0j}^{(k)} + \vec{\Psi}^T \vec{q}_j^{(k)}}{b_j - L_j^{(k)}} \right)}_{\mathcal{L}_j(b_j,\vec{\Psi})}$$

or:

$$\mathcal{L}(\vec{b},\vec{\Psi}) = r_0^{(k)} + \vec{\Psi}^T \vec{r}_{con}^{(k)} + \sum_{j=1}^{N} \mathcal{L}_j(b_j,\vec{\Psi})$$

Due to the KKT conditions:

$$\Psi_i \geq 0\,, \quad i = 1, \ldots, M_c$$

Solve it using a Lagrangian Function

Separable Problem: Min L_i & find the corresponding b_i:

$$A_j(\vec{\Psi}) = \left(p_{0j}^{(k)} + \vec{\Psi}^T \vec{p}_j^{(k)}\right)^{\frac{1}{2}}, \qquad B_j(\vec{\Psi}) = \left(q_{0j}^{(k)} + \vec{\Psi}^T \vec{q}_j^{(k)}\right)^{\frac{1}{2}}, \quad i = 1, \ldots, M_c$$

$$b_j(\vec{\Psi}) = \frac{A_j L_j^{(k)} + B_j U_j^{(k)}}{A_j + B_j}$$

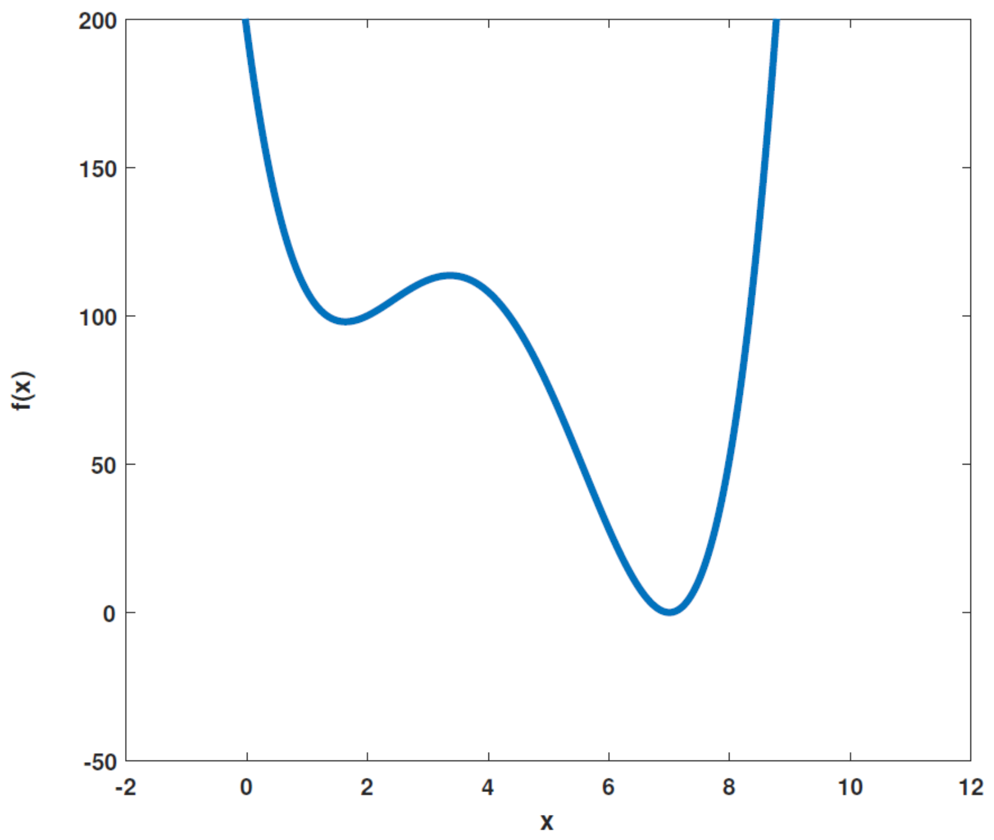
$$\mathcal{L}(\vec{\Psi}) = r_0^{(k)} + \vec{\Psi}^T \vec{r}_{con}^{(k)} + \sum_{j=1}^{N} \left(\frac{A_j^2}{U_j^{(k)} - b_j(\vec{\Psi})} + \frac{B_j^2}{b_j(\vec{\Psi}) - L_j^{(k)}} \right)$$

Simple Example (MMA-1/7)

Minimise: $f_0 = \left((x-1)^2+3\right)(x-7)^2$

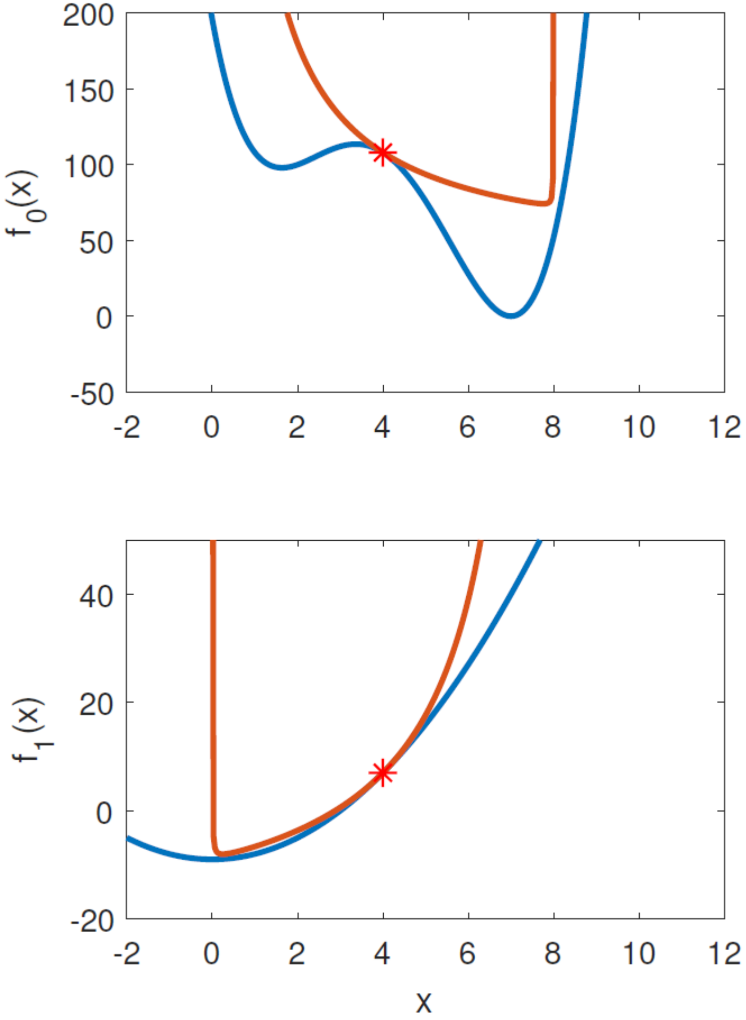
subject to: $f_1 = x^2 - 9 \leq 0$

and: $0 \leq x \leq 8$

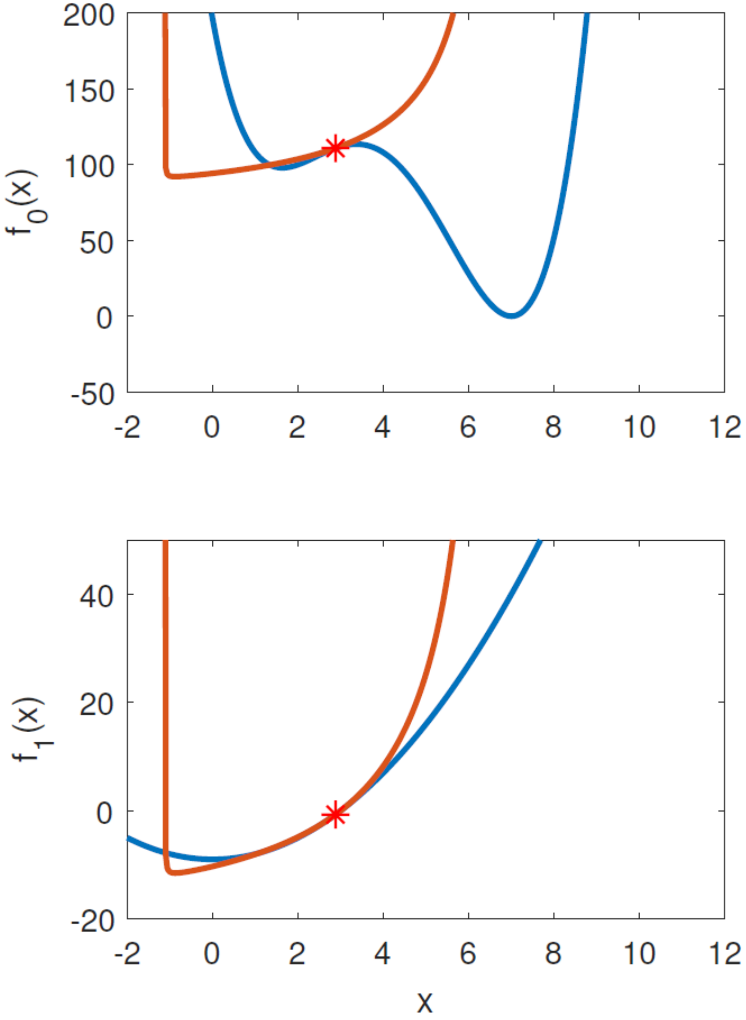


Analytical Solution: $x = 1.6340$

Simple Example (MMA-2/7)

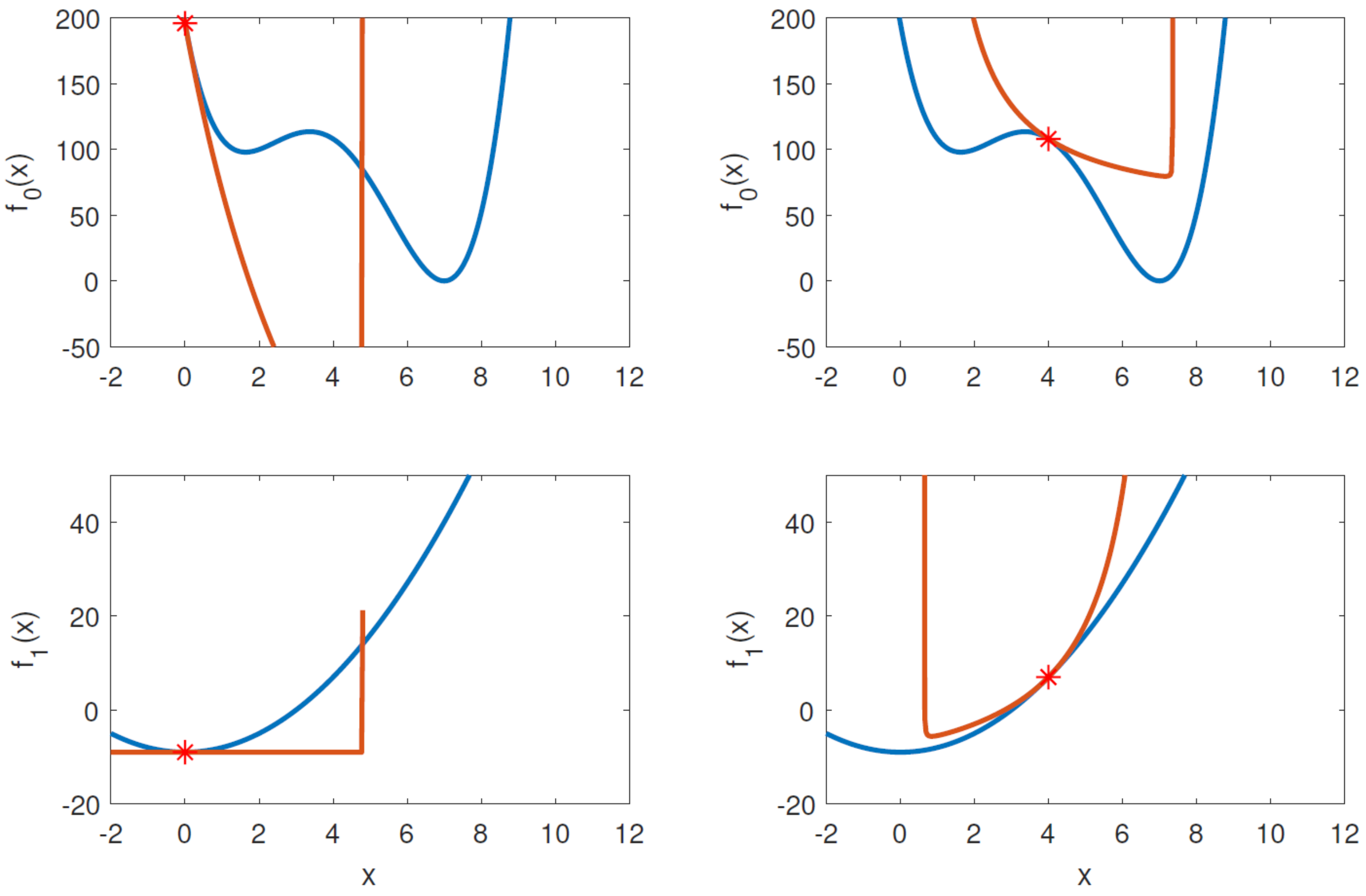


Step 0



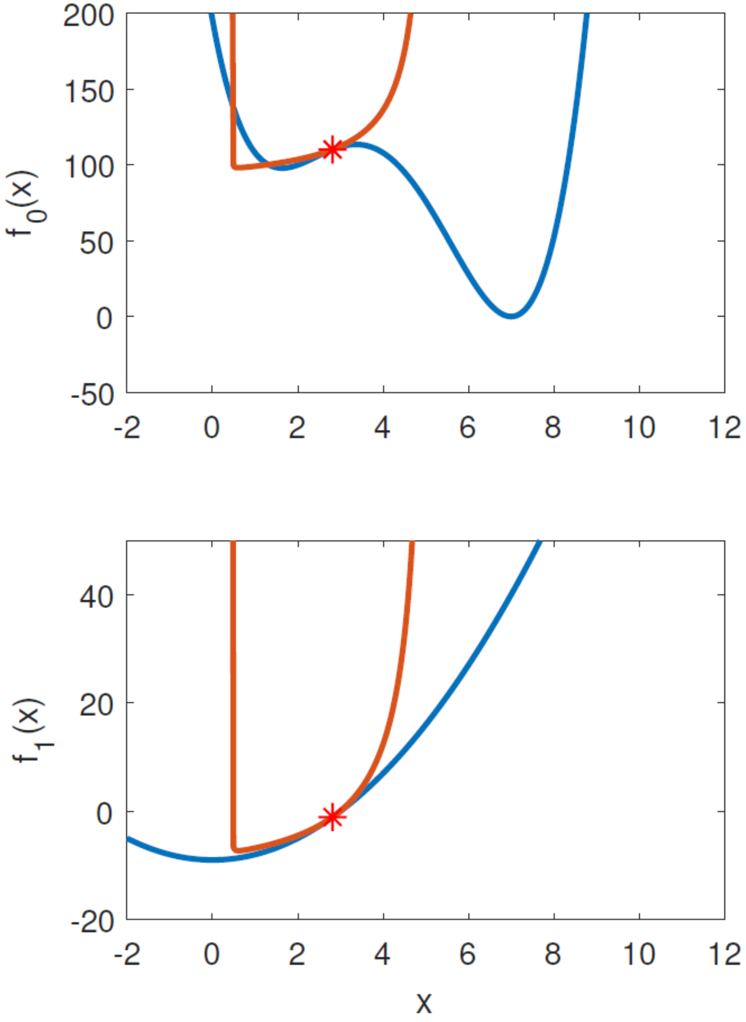
Step 1

Simple Example (MMA-3/7)

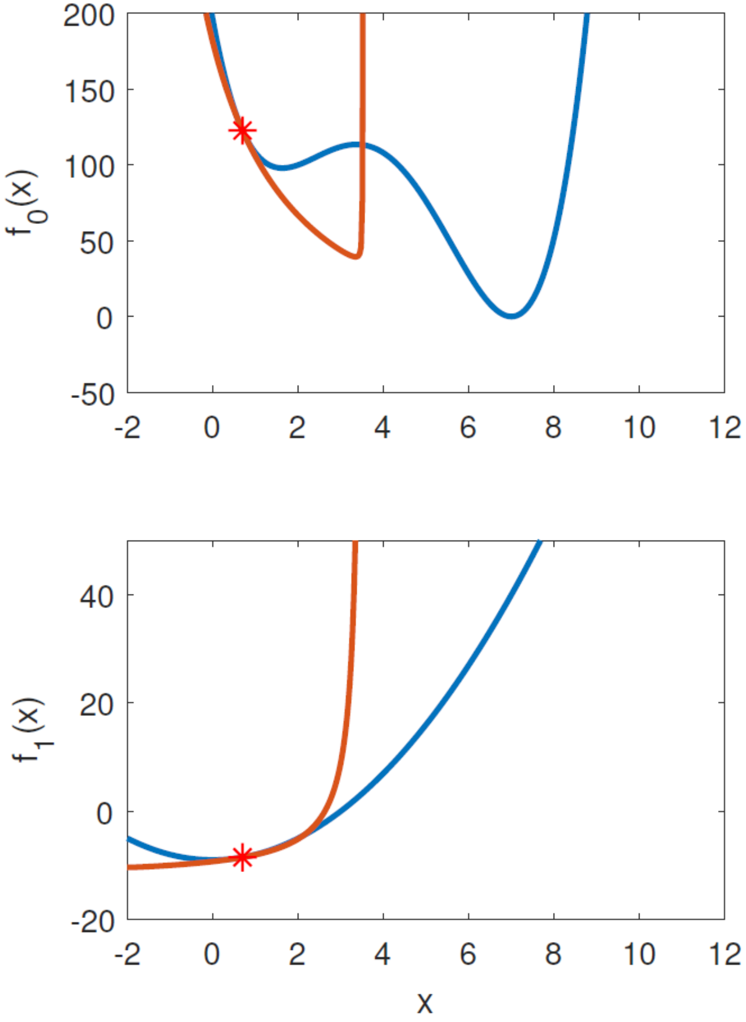


Step 2 **Step 3**

Simple Example (MMA-4/7)

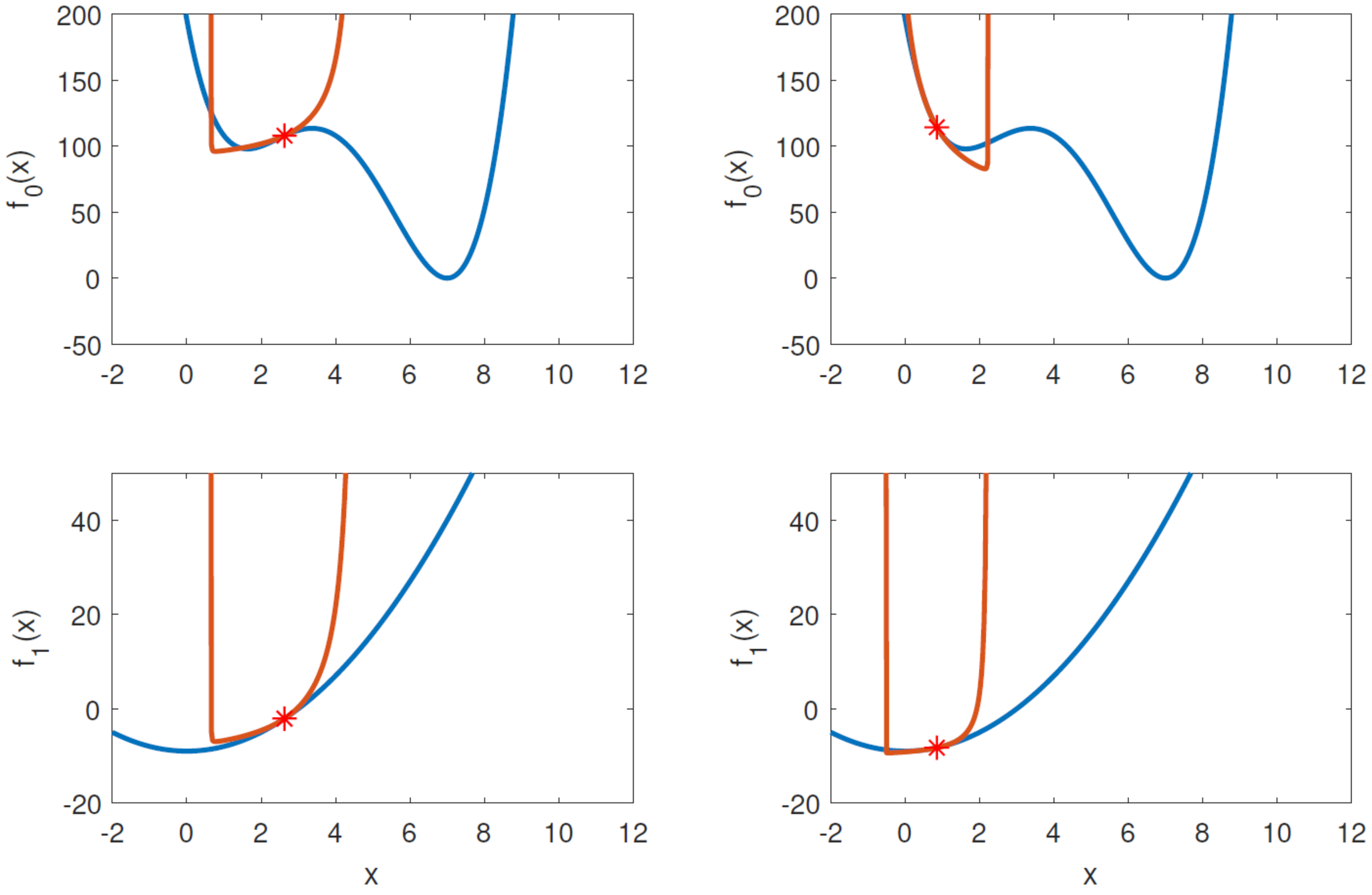


Step 4



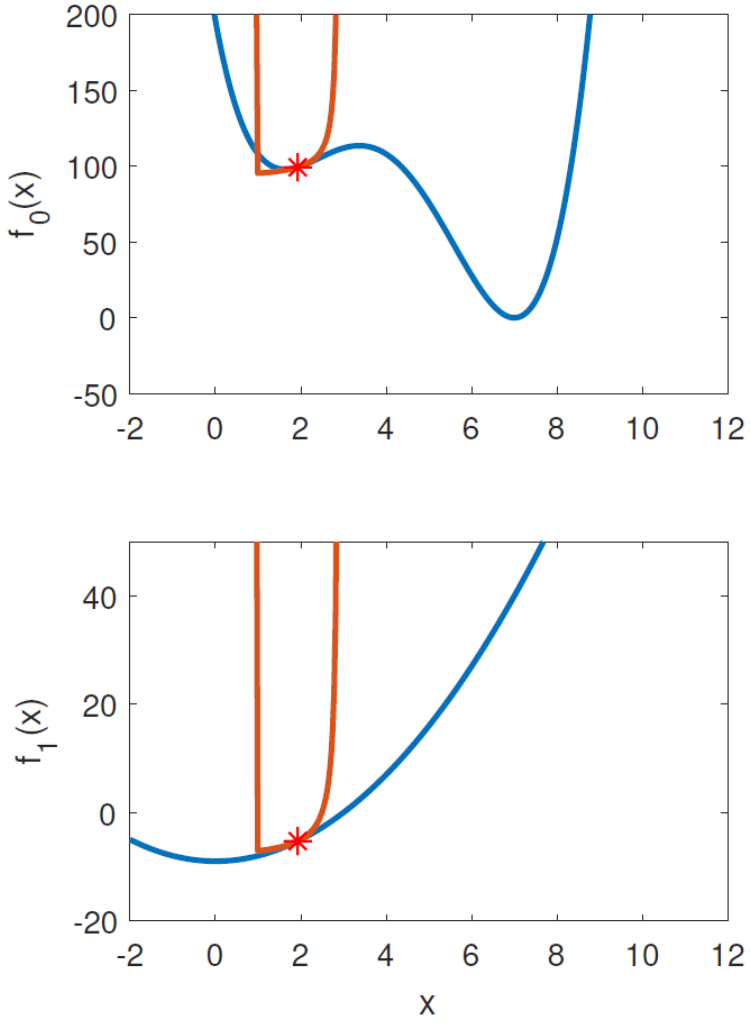
Step 5

Simple Example (MMA-5/7)

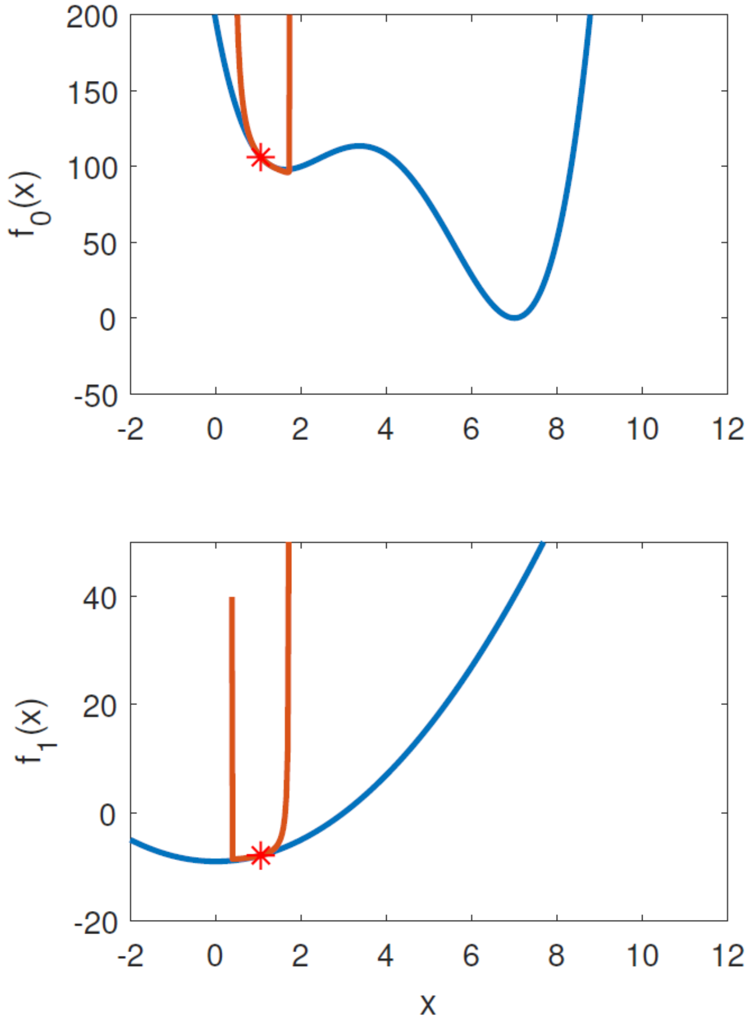


Step 6 **Step 7**

Simple Example (MMA-6/7)



Step 8



Step 9

Simple Example (MMA-7/7)

Step	x	f_0	f_1	L	U	α	β	$\hat{x}$
0	4	108	7	0	8	0.4	7.6	2.88
1	2.88	110.90	-0.71	-1.12	6.88	0	6.48	$\mathbf{3 \cdot 10^{-8}}$
2	$\mathbf{3 \cdot 10^{-8}}$	196	-9	-4.8	4.8	0	4	4
3	4	108	7	0.64	7.36	0.98	7.02	2.82
4	2.82	110.24	-1.07	0.46	5.17	0.70	4.93	0.70
5	0.70	122.74	-8.51	-2.12	3.52	0	3.24	2.63
6	2.63	107.99	-2.1	0.65	4.60	0.85	4.41	0.85
7	0.85	114.38	-8.28	-0.53	2.23	0	2.09	1.93
8	1.93	99.31	-5.29	0.96	2.89	1.05	2.80	1.05
9	1.05	106.14	-7.89	0.38	1.73	0.45	1.66	1.63

Read more in:

- Svanberg, K.: The method of moving asymptotes - a new method for structural optimization. International Journal for Numerical Methods in Engineering, 24:359{373, 1987.
- Svanberg, K. and Werme, M.: Topology optimization by a neighbourhood search method based on efficient sensitivity calculations. International Journal for Numerical Methods in Engineering, 67(12):1670{1699, 2006.
- Svanberg, K.: "A class of globally convergent optimization methods based on conservative convex separable approximations", SIAM Journal of Optimization, Vol. 12, No. 2, pp. 555–573, 2002.
- Svanberg, K. : "The method of moving asymptotes – Modelling aspects and solution schemes", Lecture notes for the DCAMM course". Advancecl Topics in Structural Optimization, Lyngby, June 25 - July 3, 1998.
- Fleury, C.: "Structural weight optimization by dual methods of convex programming", International Journal for Numerical Methods in Engineering 14, 1761-1783, 1979.
- Fleury, C. and Braibant, V. : "Structural optimization - a new dual method using mixed variables", International Journal for Numerical Methods in Engineering 23, 409-428, 1986. *(this is a special case of MMA, resulting if L=0 and U→∞)*
- Fletcher, R. : "Practical Methods of Optimization", Second edition, John Wiley & Sons. 1987.