BASIC ITERATIVE METHODS

For $A\vec{u} = \vec{f}$: with $A = D - L - U$

Jacobi:

$$\vec{v}^{n+1} = \underbrace{D^{-1}(L+U)}_{\text{Jacobi iteration matrix}} \vec{v}^{n} + D^{-1}\vec{f}$$

$$G_J = D^{-1}(L+U)$$

$$\Rightarrow \boxed{\vec{v}^{n+1} = G_J \vec{v}^{n} + D^{-1} f}$$

Jacobi with Relaxation:

$$\left.\begin{aligned} \vec{v}^{*} &= G_J \vec{v}^{n} + D^{-1}\vec{f} \\ \vec{v}^{n+1} &= \omega \vec{v}^{*} + (1-\omega)\vec{v}^{n} \end{aligned}\right\} \quad \boxed{\vec{v}^{n+1} = \underbrace{[(1-\omega)I + \omega G_J]}_{G_{J\omega} = (1-\omega)I + \omega G_J} \vec{v}^{n} + \omega D^{-1}\vec{f}}$$

The same in Delta Formulation:

$$\vec{v}^{n+1} = (1-\omega)\vec{v}^{n} + \omega D^{-1}[(L+U)\vec{v}^{n} + \vec{f}] = \vec{v}^{n} + \omega D^{-1}[\underbrace{(-D+L+U)\vec{v}^{n} + \vec{f}}_{r^{n}}]$$

$$\Rightarrow \boxed{\vec{v}^{n+1} = \vec{v}^{n} + \omega D^{-1}\vec{r}^{n}}$$

Gauss-Seidel:

$$(D-L)\,\vec{v}^{\,n+1} = U\vec{v}^{\,n} + \vec{f} \Rightarrow$$

$$\Rightarrow \boxed{\vec{v}^{\,n+1} = (D-L)^{-1}U\vec{v}^{\,n} + (D-L)^{-1}\vec{f}}$$

Gauss-Seidel with Relaxation:

you may develop it by yourself!

Comments:

Should we inverse $(D-L)^{-1}$?
Explicit & Point-Implicit variants
Parallelization (Jacobi or Gauss-Seidel?)

General Iterative Scheme:

$$\boxed{\vec{v}^{n+1} = G\vec{v}^{n} + \vec{b}} \quad \acute{\eta} \quad \boxed{\mathbb{M}\vec{v}^{n+1} = \mathbb{N}\vec{v}^{n} + \vec{f}}$$

so: $\mathbb{G} = \mathbb{M}^{-1}\mathbb{N}$, $\vec{b} = \mathbb{M}^{-1}\vec{f}$, $\underline{\underline{\mathbb{A} = \mathbb{M} - \mathbb{N}}}$

Two Questions:

(1) If the iterative scheme converges to any value, is this the real solution to the system of equations to be solved?
(2) Under which conditions such an iterative scheme does converge?

Answering Question (1):

Assume that this converged to $\vec{\underset{\sim}{v}}$.

$$M\vec{\underset{\sim}{v}} = N\vec{\underset{\sim}{v}} + \vec{f} \Rightarrow$$

$$\Rightarrow (M-N)\vec{\underset{\sim}{v}} = \vec{f} \Rightarrow$$

$$\Rightarrow \quad A\vec{\underset{\sim}{v}} = \vec{f}$$

so: $\vec{\underset{\sim}{v}} \equiv \vec{u}$

Answering Question (2):

$$\vec{v}^{\,n+1} = G\vec{v}^{\,n} + \vec{b}$$

(exact) $$\vec{u} = G\vec{u} + \vec{b}$$

$$\boxed{\vec{e}^{\,n+1} = G\vec{e}^{\,n}}$$

$$\left.\begin{aligned}\vec{e}^{\,n+1} &= G\vec{e}^{\,n}\\ \vec{e}^{\,n} &= G\vec{e}^{\,n-1}\\ &\vdots\\ \vec{e}^{\,1} &= G\vec{e}^{\,0}\end{aligned}\right\} \Rightarrow \boxed{\vec{e}^{\,n+1} = [G]^{n+1}\vec{e}^{\,0}}$$

Spectral Radius (G):

$$\rho(G) = \max\left(|\lambda|,\ \lambda \in \lambda(G)\right)$$

$\rightarrow$ **If ρ(G)<1 this iterative scheme converges to the right solution**

Two Relevant Theorems:

Theorem 1: Math. series G^n (i.e. $(I, G, G^2, G^3 \ldots)$) converges to zero, if and only if: $\underline{\underline{\rho(G) < 1}}$

Theorem 2: Sum $\sum_{k=0}^{\infty} G^k$ converges if and only if $\underline{\underline{\rho(G) < 1}}$.

Under these conditions, the following

- $(I - G)$ is an invertible matrix
- $\sum_{k=0}^{\infty} G^k$ converges to $(I - G)^{-1}$.

Why Series G^n is of interest?

$$\vec{e}^{\,n+1} = [G]^{n+1}\,\vec{e}^{\,0}$$

Theorem 1: Proof of the Necessary Condition:

$$\text{αν} \quad G^n \to \emptyset \quad \Rightarrow \quad \rho(G) < 1$$

Let $\lambda_1 =$ be the max. eigenvalue of **G**

$\vec{q}_1 =$ corresponding eigenvector (unit)

$$\left.\begin{array}{l}\lambda_1 \\ \vec{q}_1\end{array}\right\} \quad G\vec{q}_1 = \lambda_1 \vec{q}_1 \Rightarrow$$

$$\Rightarrow G^2\vec{q}_1 = G\lambda_1\vec{q}_1 = \lambda_1 \underbrace{G\vec{q}_1}_{\lambda_1\vec{q}_1} = \lambda_1^2\vec{q}_1$$

$$G^n\vec{q}_1 = \lambda_1^n\vec{q}_1 \Rightarrow \|G^n\vec{q}_1\|_2 = \|\lambda_1^n\vec{q}_1\|_2 = |\lambda_1^n|\,\|\vec{q}_1\|_2 = |\lambda^n|$$

thus, for $\|G^n\| \to \emptyset$ **it should** $\rho(G) < 1$

From Theorem 2: only its second part

$$\text{OTL} \quad \sum_{K=0}^{\infty} G^K \longrightarrow (I-G)^{-1}$$

Identity: $(I-G^{K+1}) = (I-G)(I+G+G^2+\cdots G^K)$

Since $\rho(G)<1$ it is impossible that $G\vec{q}=\vec{q} \Rightarrow (I-G)\vec{q}=0$ (else $\lambda=1$!)

$\Rightarrow (I-G) =$ invertible

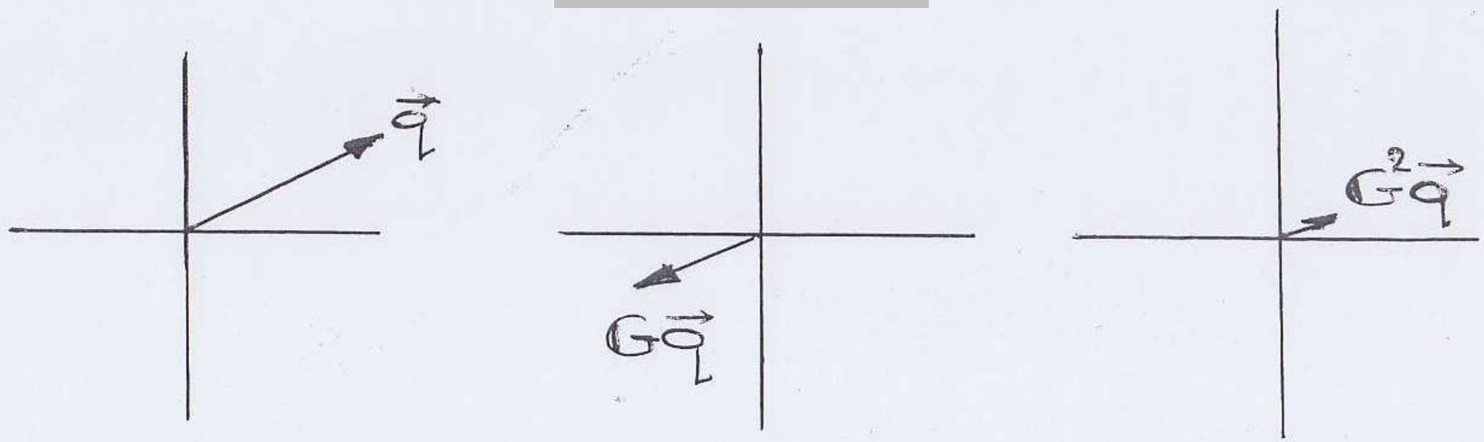
$$\Rightarrow \underbrace{(I-G)^{-1}(I-G^{K+1})}_{\text{From Theorem 1, this converges to: } (I-G)^{-1} I = (I-G)^{-1}} = I+G+G^2+\cdots+G^K$$

Eigenproblem vs. Convergence:

$\vec{q}$ = eigenvector of G

λ = eigenvalue of G

$$\boxed{G\vec{q} = \lambda\vec{q}}$$



when $|\lambda_i| < 1$ $(\forall i)$ τότε $G^n \vec{e}^{\,0} = \lambda_i^n \vec{e}^{\,0} \rightarrow \emptyset$ when $n \rightarrow \infty$

Apply G to any: $\vec{v}$

$\vec{v}$ analysis in a vector basis $(\vec{q}_1, \vec{q}_2, \ldots, \vec{q}_N)$

If $|\lambda_i| < 1, \ \forall i$, then $G\vec{v} \longrightarrow \emptyset$

Practically:

- Propose an Iterative scheme $\vec{v}^{\,n+1} = G\vec{v}^{\,n} + \vec{b}$
- Thus, I am proposing G, for a splitting $A = M - N$ with $G = M^{-1}N$.
- To converge, it should $\rho(G) < 1$.
- The smaller, the faster to converge!!
- Converge does not depend on initialization $\vec{v}^{\,0}$

Asymptotic Convergence Factor

$\rho(G)$

this is the worst factor with which the error reduces from iteration to iteration

Asymptotic Convergence Rate

$$-\log_{10}(\rho(G))$$

It is $\vec{e}^{\,n} = [G]^n \vec{e}^{\,0}$ and $\|\vec{e}^{\,n}\| \leq \|G\|^n \cdot \|\vec{e}^{\,0}\| \Rightarrow \dfrac{\|\vec{e}^{\,n}\|}{\|\vec{e}^{\,0}\|} \leq \|G\|^n$

Assume that we want to reduce the error by d orders of magnitude:

$$\frac{\|\vec{e}^{\,n}\|}{\|\vec{e}^{\,0}\|} \leq 10^{-d}$$

Assume that we want to reduce the error by d orders of magnitude:

$$(\rho[G])^n \leq 10^{-d} \Rightarrow \boxed{n \geq \frac{d}{-\log_{10}(\rho(G))}}$$

Application (Attention NEW Symbols)

We are solving $A\vec{x}=\vec{b}$ using the iterative scheme

$$\vec{x}^{k+1} = \vec{x}^{k} + a(\vec{b}-A\vec{x}^{k}), \quad a \geq 0$$

or

$$\vec{x}^{k+1} = \underbrace{(I-aA)}\vec{x}^{k} + a\vec{b}$$

ITERATION MATRIX $G_a = I - aA$

Convergence depends on $\rho(G_a)=\rho(I-aA)$

Let λ_i be the real eigenvalues of A

$$\lambda_{min} \leq \lambda_i \leq \lambda_{max}$$

Then, the eigenvalues of G_a (μ_i)

$$\underbrace{1-a\lambda_{max}}_{\text{SIGNED!}} \leq \mu_i \leq \underbrace{1-a\lambda_{min}}_{\text{SIGNED!}}$$

Cases

(a) $\lambda_{min} < 0$, $\lambda_{max} > 0$

$$\Rightarrow \max(\mu_i) = 1 - a\lambda_{min} > 1$$

the method diverges $\forall a > 0$

(β) $\underline{\lambda_i > 0 , \forall i} \Rightarrow \lambda_{min} > 0$

To converge, it should $\begin{cases} 1 - a\lambda_{min} < 1 \;\&\; 1 - a\lambda_{min} > -1 \\ 1 - a\lambda_{max} > -1 \;\&\; 1 - a\lambda_{max} < 1 \end{cases}$

$$1 - a\lambda_{min} < 1 \xrightarrow{\lambda_{min} > 0} a > 0 \quad \checkmark$$

or

$$\left.\begin{aligned} 1 - a\lambda_{min} > -1 &\Longrightarrow a < 2/\lambda_{min} \\ 1 - a\lambda_{max} > -1 &\Longrightarrow a < 2/\lambda_{max} \end{aligned}\right\} \text{κυριαρχεί η } a < \frac{2}{\lambda_{max}}$$

$$1 - a\lambda_{max} < 1 \Longrightarrow a > 0 \quad \checkmark$$

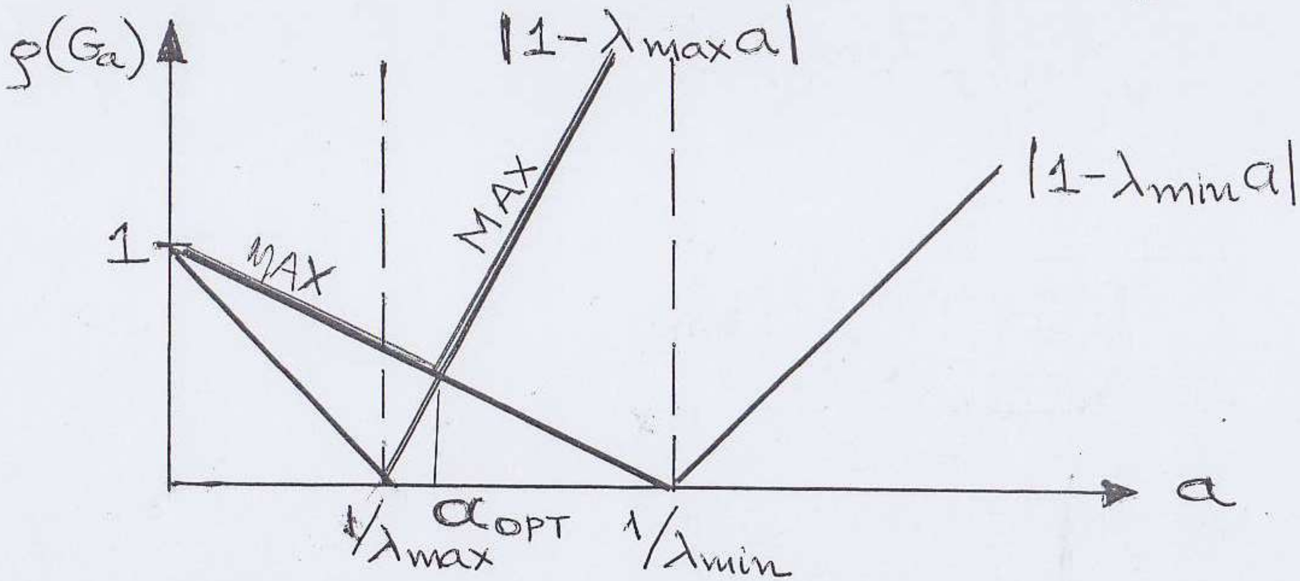
So, the method converges if

$$\boxed{0 < a < \frac{2}{\lambda_{max}}}$$

Optimal value of a: (a_{OPT})

$$a_{OPT} = \arg\min_{a} \rho(G_a)$$

$$\rho(G_a) = \max\left\{ |1 - a\lambda_{min}|, |1 - a\lambda_{max}| \right\}$$



$$-1 + \lambda_{max} a_{OPT} = 1 - \lambda_{min} a_{opt} \Rightarrow a_{opt} = \frac{2}{\lambda_{min} + \lambda_{max}}$$

$$\rho_{opt}(G_a) = 1 - \frac{2\lambda_{min}}{\lambda_{min} + \lambda_{max}} = \frac{\lambda_{max} - \lambda_{min}}{\lambda_{max} + \lambda_{min}}$$

Useful Definitions

Non-Negative Matrix

(a) $A \geqslant 0$, OTAN $\forall\ a_{ij} \geqslant 0$

(β) $A \leq B \Rightarrow a_{ij} \leq \beta_{ij}$, $\forall (i,j)$

(γ) Positive Matrix $A > 0$, iff $\forall\ a_{ij} > 0$

Reduced Matrix

A is reduced matrix if there is a matrix P such as that

$PAP^T =$ upper-triangular

Theorem 3:

If G is a non-negative matrix, a necessary and sufficient condition for ρ(G)<1 is the following conditions:

$$\left\| \begin{array}{l} (I-G) = \text{invertible} \\ (I-G)^{-1} = \text{non-negative} \end{array} \right.$$

Proof (of Sufficient condition):

αν $\rho(G)<1$, Θεώρημα 2 $\Rightarrow$

$\Rightarrow$ (α) $I-G$ = ΟΜΑΛΟ

(β) $(I-G)^{-1} = \lim \sum_{k=0}^{\infty} G^k$

Ομως $G>0 \Rightarrow \sum_k G^k > 0 \Rightarrow (I-G)^{-1} =$ non-negative

M-Matrix: Definition

A is an M-matrix if it satisfies the following conditions:

(α) $a_{i,i} > 0 \quad , \ \forall i$

(β) $a_{i,j} \leq 0 \quad , \ \forall (i,j) \ , i \neq j$

(γ) $A =$ invertible

(δ) $A^{-1} \geq 0$

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(α) $a_{i,i} > 0 \quad , \ \forall i$

(β) $a_{i,j} \leq 0 \quad , \ \forall (i,j) \ , i \neq j$

(γ) $\rho(B) < 1$ where $B = I - D^{-1} A$

M-Matrix: Why these are of interest to us?

JACOBI

$$D\vec{v}^{n+1} = (D-A)\vec{v}^{n} + \vec{f}$$

$$\vec{v}^{n+1} = (I-D^{-1}A)\vec{v}^{n} + D^{-1}\vec{f}$$

$$\vec{v}^{n+1} = G\vec{v}^{n} + \vec{b} \quad , \quad G = I - D^{-1}A$$

It was shown that if A=M-matrix $\Rightarrow \rho(G) < 1$

- For the iterative scheme to converge for any initial condition, it should

$$\rho(G) < 1$$

which is ensured if A=M-matrix
(A results from the discretization of the equation)

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REGULAR SPLITTING OF A MATRIX

Definition: A=M-N=regular splitting if M^{-1} and N are non-negative matrices

THEOREM 4: if M,N= a regular splitting of A, then $\rho(M^{-1}N)<1$, if and only if

$A =$ invertible & $A^{-1} =$ non-negative

Lesson Learned:

$$M\vec{v}^{n+1} = N\vec{v}^{n} + \vec{f} \Rightarrow \vec{v}^{n+1} = M^{-1}N\vec{v}^{n} + M^{-1}\vec{f}$$

This iterative scheme converge always if M and N correspond to a regular splitting of A
& A=M-matrix

PRECONDITIONING

$$\vec{v}^{\,n+1} = G\vec{v}^{\,n} + \vec{b}$$

$$G = M^{-1}N \quad , \quad \vec{b} = M^{-1}\vec{f}$$

LEFT PRECONDITIONING:

$$P^{-1}A\vec{u} = P^{-1}\vec{f}$$

$\mathbb{P}$ = ΕΥΚΟΛΑ ΑΝΤΙΣΤΡΕΨΙΜΗ ΠΡΟΣΕΓΓΙΣΗ ΤΟΥ A

RIGHT PRECONDITIONING:

$$AQ(Q^{-1}\vec{u}) = \vec{f}$$

Real Residual....