Grid Generation

IPSP Computational Mechanics, NTUA
Formula Sheet (for Use in the Exams)
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STRUCTURED (CURVILINEAR BODY-FITTED) GRIDS (POLAR GRID)

Covariant Vector Basis (v.b.): At each point of the structured grid, on the (x, y) plane, this is defined using the following vector bases

$$\overrightarrow{g}_1 = \frac{\partial \overrightarrow{r}}{\partial \xi} = (x_\xi, y_\xi), \qquad \overrightarrow{g}_2 = \frac{\partial \overrightarrow{r}}{\partial \eta} = (x_\eta, y_\eta)$$

The covariant second–order metrics are defined as

$$g_{ij} = \overrightarrow{g}_i \cdot \overrightarrow{g}_j$$

namely

$$g_{11} = x_\xi^2 + y_\xi^2, \quad g_{22} = x_\eta^2 + y_\eta^2, \quad g_{12} = g_{21} = x_\xi x_\eta + y_\xi y_\eta$$

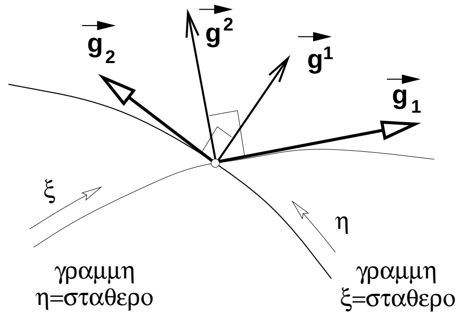
Any vector $\overrightarrow{A}$ can be analyzed on this vector basis as

$$\overrightarrow{A} = A^i \overrightarrow{g}_i$$

where A^i are its contravariant components. $\overrightarrow{r}$ is the position vector.

Contravariant Vector Basis: This is defined by the vector basis

$$\overrightarrow{g}^1 = \nabla \xi = (\xi_x, \xi_y), \qquad \overrightarrow{g}^2 = \nabla \eta = (\eta_x, \eta_y)$$



The contravariant second–order metrics are defined as

$$g^{ij} = \overrightarrow{g}^i \cdot \overrightarrow{g}^j$$

namely

$$g^{11} = \xi_x^2 + \xi_y^2, \quad g^{22} = \eta_x^2 + \eta_y^2, \quad g^{12} = g^{21} = \xi_x \eta_x + \xi_y \eta_y.$$

Any vector $\overrightarrow{A}$ can be analyzed on this vector basis as

$$\overrightarrow{A} = A_i \overrightarrow{g}^i$$

where A_i are its covariant components.

Fundamental Relationships: It is

$$\overrightarrow{g}_i \cdot \overrightarrow{g}^j = \delta_i^j$$

where δ_i^j is the Kronecker symbol. From this expression, it can be proved that $A_i = \overrightarrow{A} \cdot \overrightarrow{g}_i$ and $A^i = \overrightarrow{A} \cdot \overrightarrow{g}^i$. If

$$J = x_\xi y_\eta - x_\eta y_\xi$$

is the Jacobian of the transformation $(x, y) \leftrightarrow (\xi, \eta)$, then

$$\xi_x = \frac{y_\eta}{J}, \quad \xi_y = \frac{-x_\eta}{J}, \quad \eta_x = \frac{-y_\xi}{J}, \quad \eta_y = \frac{x_\xi}{J}$$

*This formula sheet is the only aid students may use during their examination on the relevant material. There may be typos. It is each student's responsibility to check it against the textbook and their notes. The instructor kindly asks to be informed of any corrections.

Similarly

$$g^{11}=\frac{g_{22}}{J^2}, \quad g^{22}=\frac{g_{11}}{J^2}, \quad g^{12}=\frac{-g_{12}}{J^2}$$

The elementary arc-length is $ds^2 = g_{ij}d\xi^i d\xi_j$.A twice-repeated index implies summation (Einstein convention).

Differential Operators: The gradient of a scalar function Φ is written as

$$\nabla\Phi=\frac{\partial\Phi}{\partial\xi^i}\overrightarrow{g}^i$$

The divergence of a vector function $\overrightarrow{A}$ is written as

$$\nabla\cdot\overrightarrow{A}=\frac{1}{J}\frac{\partial(JA^i)}{\partial\xi^i}$$

The Laplace operator, applied to Φ, becomes

$$\nabla^2\Phi=\frac{1}{J}\frac{\partial}{\partial\xi^i}\left(Jg^{ij}\frac{\partial\Phi}{\partial\xi^j}\right)$$

where $\xi^1\equiv\xi$, $\xi^2\equiv\eta$.

Derivatives of the covariant v.b: It is

$$\frac{\partial\overrightarrow{g}_i}{\partial\xi^j}=\Gamma^k_{ij}\overrightarrow{g}_k$$

where the Christoffel symbols of the second kind are defined as

$$\Gamma^k_{ij}=\Gamma^k_{ji}=\frac{1}{2}g^{mk}\left(\frac{\partial g_{im}}{\partial\xi^j}+\frac{\partial g_{jm}}{\partial\xi^i}-\frac{\partial g_{ij}}{\partial\xi^m}\right)$$

Structured Grid generation based on Elliptic type PDE's: The following PDEs

$$\nabla^2\xi^m=f^m$$

must be solved; these are transformed to (ξ,η) as follows:

$$g^{ij}\frac{\partial^2\overrightarrow{r}}{\partial\xi^i\partial\xi^j}+f^i\frac{\partial\overrightarrow{r}}{\partial\xi^i}=0$$

Functions f^m act as grid-quality control mechanisms. For 2D grids, these equations can be developed as:

$$g_{22}x_{\xi\xi}-2g_{12}x_{\xi\eta}+g_{11}x_{\eta\eta}+J^2f^1x_\xi+J^2f^2x_\eta = 0$$
$$g_{22}y_{\xi\xi}-2g_{12}y_{\xi\eta}+g_{11}y_{\eta\eta}+J^2f^1y_\xi+J^2f^2y_\eta = 0$$

STRUCTURED SURFACE GRIDS

First Fundamental Form of the Surface: At each point on the surface which is associated with a parameterization (ξ,η), the first fundamental form of the surface (I) is defined as

$$I=d\overrightarrow{r}\cdot d\overrightarrow{r}$$

It is

$$I=Ed\xi^2+2Fd\xi d\eta+Gd\eta^2=g_{ij}d\xi^i d\xi^j$$

where

$$E=g_{11}=\overrightarrow{r}_\xi\cdot\overrightarrow{r}_\xi$$
$$F=g_{12}=g_{21}=\overrightarrow{r}_\xi\cdot\overrightarrow{r}_\eta$$
$$G=g_{22}=\overrightarrow{r}_\eta\cdot\overrightarrow{r}_\eta$$

are the first fundamental coefficients of the surface at this point. Quantity I is invariant, thus independent of the parameterization (ξ,η). However, the first fundamental coefficients of the surface (or the covariant second-order metrics) depend on the parametrization. The surface Jacobian is expressed in terms of them as

$$J_s=\sqrt{EG-F^2}=\sqrt{g_{11}g_{22}-g_{12}^2}$$

Unit Normal Vector to the Surface: At each point on the surface which is associated with the parameterization (ξ,η), the unit normal vector is

$$\overrightarrow{N}=\frac{\overrightarrow{r}_\xi\times\overrightarrow{r}_\eta}{|\overrightarrow{r}_\xi\times\overrightarrow{r}_\eta|}$$

✎ **Second Fundamental Form of the Surface:** At each point on the surface which is associated with a parameterization (ξ,η), the second fundamental form of the surface is defined (II) as

$$II = -d\overrightarrow{r}\cdot d\overrightarrow{N} = d^2\overrightarrow{r}\cdot d\overrightarrow{N}$$

it is

$$II = Ld\xi^2 + 2Md\xi d\eta + Nd\eta^2 = \Omega_{ij}d\xi^i d\xi^j$$

where

$$\begin{aligned} L &= \Omega_{11} = -\overrightarrow{r}_{\xi}\cdot\overrightarrow{N}_{\xi} = \overrightarrow{r}_{\xi\xi}\cdot\overrightarrow{N} \\ M &= \Omega_{12} = \Omega_{21} = \frac{-1}{2}\left(\overrightarrow{r}_{\xi}\cdot\overrightarrow{N}_{\eta} + \overrightarrow{r}_{\eta}\cdot\overrightarrow{N}_{\xi}\right) = \overrightarrow{r}_{\xi\eta}\cdot\overrightarrow{N} \\ N &= \Omega_{22} = -\overrightarrow{r}_{\eta}\cdot\overrightarrow{N}_{\eta} = \overrightarrow{r}_{\eta\eta}\cdot\overrightarrow{N} \end{aligned}$$

are the second fundamental coefficients of the surface at this point. Quantity II is invariant, thus independent of the parameterization (ξ,η). However, the second fundamental coefficients of the surface depend on the parametrization.

✎ **Curvatures:** If C is a curve on a surface S, associated with the parametrization (ξ,η), the normal curvature vector $\overrightarrow{k}_n$ at a point P of C is given by

$$\overrightarrow{k}_n = (\overrightarrow{k}\cdot\overrightarrow{N})\overrightarrow{N}$$

where $\overrightarrow{k}$ is the curvature vector of C at P.

The normal curvature κ_n is defined as

$$\kappa_n = \overrightarrow{k}\cdot\overrightarrow{N} = \frac{II}{I}$$

and, as the ratio of two invariant quantities, is also invariant.

All curves on surface S which pass by point P and are, there, tangent to the same staright line, have the same normal curvature value.

The normal section of a surface at point P is every curve of S which corresponds to its section by a plane that includes the normal unit vector $\overrightarrow{N}$ at P.

The curvature of a normal section of S at P is equal to the normal curvature at P, which is invariant, for the said cross–section.

The principal directions are those $d\xi : d\eta$ at P at which κ_n becomes maximum and minimum. The corresponding normal curvatures are referred to as the principal curvatures. These, denoted by κ (κ_1 and κ_2) are the roots of equation

$$\left(EG-F^2\right)\kappa^2 - (EN+GL-2FM)\kappa + \left(LN-M^2\right) = 0$$

The mean curvature μ at a point P os a surface S is equal to half of the sum of the local principal directions,

$$\mu = \frac{1}{2}(\kappa_1+\kappa_2) = \frac{EN+GL-2FM}{2(EG-F^2)} = \frac{1}{2}g^{ij}\Omega_{ij}$$

where the contravariant second-order metrics are related to the covariant ones through the following relationships

$$g^{11} = \frac{g_{22}}{J_s^2}, \qquad g^{12} = -\frac{g_{12}}{J_s^2}, \qquad g^{22} = \frac{g_{11}}{J_s^2}$$

while

$$g_{ik}g^{kj} = \delta_i^j$$

The Gaussian curvature K as a point P os surface S is equal to the product of the local principal curvatures,

$$K = \kappa_1\kappa_2 = \frac{LN-M^2}{EG-F^2} = \frac{\Omega_{11}\Omega_{22}-\Omega_{12}^2}{g_{11}g_{22}-g_{12}^2}$$

At each point of the surface, the mean curvature and the Gaussian curvature are both invariant.

✎ **Gauss-Weingarten Equations:** Gauss Equations:

$$\begin{aligned} \overrightarrow{r}_{\xi\xi} &= \Gamma_{11}^1\overrightarrow{r}_{\xi} + \Gamma_{11}^2\overrightarrow{r}_{\eta} + \Omega_{11}\overrightarrow{N} \\ \overrightarrow{r}_{\xi\eta} &= \Gamma_{12}^1\overrightarrow{r}_{\xi} + \Gamma_{12}^2\overrightarrow{r}_{\eta} + \Omega_{12}\overrightarrow{N} \\ \overrightarrow{r}_{\eta\eta} &= \Gamma_{22}^1\overrightarrow{r}_{\xi} + \Gamma_{22}^2\overrightarrow{r}_{\eta} + \Omega_{22}\overrightarrow{N} \end{aligned}$$

or

$$\overrightarrow{r}_{,ij} = \Gamma_{ij}^k\overrightarrow{r}_{,k} + \Omega_{ij}\overrightarrow{N}$$

where Γ^k_{ij} are the surface Christoffel symbols of the second kind. These are:

$$\begin{aligned}
\Gamma^1_{11} &= \frac{GE_\xi - 2FF_\xi + FE_\eta}{2(EG-F^2)} \\
\Gamma^2_{11} &= \frac{2EF_\xi - EE_\eta + FE_\xi}{2(EG-F^2)} \\
\Gamma^1_{12} &= \frac{GE_\eta - FG_\xi}{2(EG-F^2)} \\
\Gamma^2_{12} &= \frac{EG_\xi - FE_\eta}{2(EG-F^2)} \\
\Gamma^1_{22} &= \frac{2GF_\eta - GG_\xi + FG_\eta}{2(EG-F^2)} \\
\Gamma^2_{22} &= \frac{EG_\eta - 2FF_\eta + FG_\xi}{2(EG-F^2)}
\end{aligned}$$

or

$$\Gamma^k_{ij} = \Gamma^k_{ji} = \frac{1}{2} g^{mk} \left(\frac{\partial g_{im}}{\partial \xi^j} + \frac{\partial g_{jm}}{\partial \xi^i} - \frac{\partial g_{ij}}{\partial \xi^m} \right)$$

Weingarten Equations:

$$\begin{aligned}
\overrightarrow{N}_\xi &= \beta^1_1 \overrightarrow{r}_\xi + \beta^2_1 \overrightarrow{r}_\eta \\
\overrightarrow{N}_\eta &= \beta^1_2 \overrightarrow{r}_\xi + \beta^2_2 \overrightarrow{r}_\eta
\end{aligned}$$

or

$$\overrightarrow{N}_{,i} = \beta^j_i \overrightarrow{r}_{,j}$$

where

$$\begin{aligned}
\beta^1_1 &= \frac{MF - LG}{EG - F^2} \\
\beta^2_1 &= \frac{LF - ME}{EG - F^2} \\
\beta^1_2 &= \frac{NF - MG}{EG - F^2} \\
\beta^2_2 &= \frac{MF - NE}{EG - F^2}
\end{aligned}$$

Contraction of the Gauss equations with g^{ij} yields

$$g^{ij} \overrightarrow{r}_{,ij} - g^{ij} \Gamma^k_{ij} \overrightarrow{r}_{,k} = g^{ij} \Omega_{ij} \overrightarrow{N} = 2\mu \overrightarrow{N}$$

✎ **Useful relationships:** It can be proved that

$$\begin{aligned}
\frac{\partial g^{i\delta}}{\partial \xi^k} &= -g^{\alpha\delta} \Gamma^i_{\alpha\kappa} - g^{\alpha i} \Gamma^\delta_{\alpha\kappa} \\
\frac{\partial J_s}{\partial \xi^i} &= J_s \Gamma^j_{ji}
\end{aligned}$$

✎ **The Beltrami Operator:** The Beltrami operator, a.k.a. as the surface Laplace operator is

$$\nabla^2_s() = \frac{1}{J_s} \frac{\partial}{\partial \xi^i} \left(J_s g^{ij} \frac{\partial()}{\partial \xi^j} \right)$$

It can be proved that

$$\nabla^2_s \xi^m = -g^{\alpha\beta} \Gamma^m_{\alpha\beta}$$

✎ **Generation of Surface Structured Grids using Elliptic type PDEs:** Requiring that

$$\nabla^2_s \xi^m = f^m$$

the following equation

$$g^{ij} \overrightarrow{r}_{,ij} + f^k \overrightarrow{r}_{,k} = 2\mu \overrightarrow{N}$$

can be derived, or

$$g^{ij} \frac{\partial^2 \overrightarrow{r}}{\partial \xi^i \partial \xi^j} + f^i \frac{\partial \overrightarrow{r}}{\partial \xi^i} = 2\mu \overrightarrow{N}$$